A Review on Semi-Analytic Methods for Solving Partial Differential Equations

Suhas Talekar[1], Kuldeep Kandwal[1], Archita Malge[2]
[1]Thakur College of Science and Commerce, Mumbai, Maharashtra, India
[2]Department of Mathematics, M. S. Bidave College of Engineering, Latur, India
Corresponding Author: Archita Malge
architamalge19@gmail.com

Abstract

Partial Differential Equations (PDEs) play a crucial role in the modeling of various physical phenomena. Exact solutions to PDEs are rare, especially for nonlinear equations. Semi-analytic methods such as the Adomian Decomposition Method (ADM), the Homotopy Analysis Method (HAM), and the Laplace Adomian Decomposition Method (LADM) provide effective alternatives for approximate analytical solutions. This review discusses the principles and methodologies of these three methods in detail and demonstrates each with an illustrative example.

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1. Introduction

Partial Differential Equations (PDEs) and Ordinary differential equations are essential mathematical tools for modeling a wide variety of physical, chemical, biological, and engineering processes involving functions of several variables [6-8]. These equations are used to describe phenomena such as fluid dynamics, electromagnetic fields, heat conduction, quantum mechanics, and population dynamics. Despite their significance, finding closed-form analytical solutions to PDEs is often difficult, especially for nonlinear or complex systems. Traditional numerical techniques like the finite element method, finite difference method, and spectral methods can provide approximate solutions, but they often suffer from high computational cost, numerical instability, and loss of qualitative information inherent in analytical expressions.

To address these limitations, researchers have developed various semi-analytical methods that combine the strengths of analytical and numerical techniques. These methods provide approximate solutions in the form of convergent series and preserve important physical characteristics of the solutions. Among these, the Adomian Decomposition Method (ADM), the Homotopy Analysis Method (HAM), and the Laplace Adomian Decomposition Method (LADM) have received significant attention due to their flexibility, convergence properties, and simplicity of implementation.

ADM, introduced by George Adomian in the early 1980s, enables the decomposition of nonlinear operators using specially constructed polynomials known as Adomian polynomials [Adomian, 1994]. HAM, developed by Shijun Liao, introduces an embedding parameter and auxiliary functions to control the convergence of the series solution without relying on small parameters [Liao, 2003]. LADM synergizes the Laplace transform with ADM to simplify computations, especially for problems involving time-dependent behavior and initial conditions [Inc, 2008].

This paper aims to provide a comprehensive review of these three semi-analytical methods, focusing on their methodologies, applications, and effectiveness in solving PDEs. Each method is accompanied by a representative example to illustrate its practical utility.

2. Adomian Decomposition Method (ADM)

The Adomian Decomposition Method (ADM) is a widely used semi-analytic technique for solving linear and nonlinear PDEs. Proposed by George Adomian in the 1980s, the method constructs the solution as a rapidly convergent infinite series without requiring linearization or small parameters [Adomian, 1994].

Methodology

Let us consider a general PDE of the form:

$$L(u) + N(u) + R(u) = g(x, t),$$

where:

- L is a linear operator,
- N is a nonlinear operator,
- R is the remainder of the linear part,
- g(x, t) is a known function.

The solution is decomposed as:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$

The nonlinear term N(u) is expressed using Adomian polynomials:

$$N(u) = \sum_{n=0}^{\infty} A_n$$

where A_n are Adomian polynomials generated from the series of u_n.

The recursive relation is developed using the inverse of the highest-order operator, typically an integral operator. This allows for successive calculation of u_n terms.

Illustrative Example:1 Heat Equation

Equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad t > 0 \tag{1}$$

Initial condition: $u(x,0) = \sin(\pi x)$

Steps:

1. Assume the solution as a series: $u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$

2. Let the zeroth component be: $u_0(x,t) = \sin(\pi x)$

3. The recursive relation is defined as: $u_{n+1}(x,t) = \int_0^t \frac{\partial^2 u_n(x,\tau)}{\partial x^2} d\tau$

4. Compute the first few components:

i. $u_1(x,t) = \int_0^t \frac{\partial^2}{\partial x^2}[\sin(\pi x)]d\tau = -\pi^2 \sin(\pi x)t$

ii. $u_2(x,t) = \int_0^t \frac{\partial^2}{\partial x^2}[-\pi^2 \sin(\pi x)\tau]d\tau = \pi^4 \sin(\pi x)\frac{t^2}{2!}$

iii. $u_3(x,t) = -\pi^6 \sin(\pi x)\frac{t^3}{3!}$

5. Therefore, the solution becomes: $u(x,t) = \sin(\pi x)\left[1 - \pi^2 t + \frac{\pi^4 t^2}{2!} - \frac{\pi^6 t^3}{3!} + \cdots\right]$

This is the Taylor expansion of the exact solution: $u(x,t) = \sin(\pi x)e^{-\pi^2 t}$ (2)

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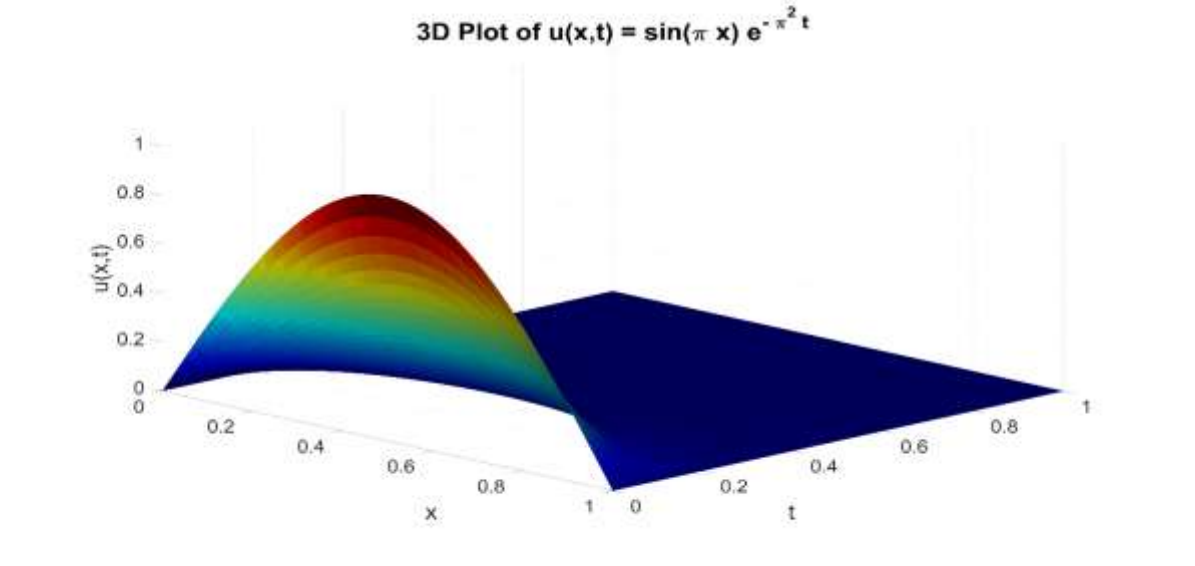

Fig:1 Solution plot of Heat Equation (2) Using MATLAB

3. Homotopy Analysis Method (HAM)

HAM is an analytic method developed by Shijun Liao in the 1990s for solving highly nonlinear problems [Liao, 2003]. Unlike perturbation methods, HAM does not rely on small parameters and provides a way to ensure the convergence of the solution series.

Methodology

The basic idea of HAM is to construct a homotopy that continuously deforms from an initial approximation to the exact solution. Consider the nonlinear PDE:

$$N\left(u\left(x,t\right)\right) = 0$$

Construct a homotopy:

$$H\left(u,\, p\right) = \left(1 - p\right) L\left[u - u_0\right] + p\, h\, N\left(u\right) = 0$$

where:

- $p \in [0, 1]$ is the embedding parameter,
- h is the convergence-control parameter,
- L is a linear operator,
- u_0 is an initial guess.

Expanding u in terms of p:

$$u(x,t;p) = u_0 + \sum_{m=1}^{\infty} u_m(x,t)p^m$$

Setting p = 1 yields the series solution:

$$u(x,t) = u_0 + \sum_{m=1}^{\infty} u_m(x,t)$$

The parameter h can be chosen to ensure convergence using h-curves.

Illustrative Example:2 Burgers' Equation

Burgers' Equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} \quad (3)$$

with the initial condition: $u(x,0) = -\sin(\pi x), \quad x \in (0,1), \quad t > 0$

Step 1: Initial Guess and Operators

- Choose the initial approximation: $u_0(x,t) = -\sin(\pi x)$
- Define:
  - Linear operator:

$$L[u] = \frac{\partial u}{\partial t}$$

  - Nonlinear operator:

$$N[u] = u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2}$$

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Step 2: Zeroth-Order Deformation Equation

Construct the zeroth-order deformation equation:

$$(1-p)L[\phi(x,t;p)-u_0(x,t)]=phN[\phi(x,t;p)]$$

where:

- $p\in[0,1]$ is the embedding parameter,
- h≠0 is the convergence-control parameter.

Step 3: Series Expansion

Assume a power series in p:

$$\phi(x,t;p)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t)p^m$$

When p=1, we obtain:

$$u(x,t)=\sum_{m=0}^{\infty}u_m(x,t)$$

Step 4: Recursive Relations

By differentiating the deformation equation with respect to pp, setting p=0p = 0, and dividing by m!m!, we get:

$$L\left[u_m\left(x,t\right)\right] \ = \ h\ R_m\left(x,t\right)$$

where R_m are the m-th order residuals computed from $u_0,u_1,\ldots,u_{m-1}$.

Examples of residual terms:

- $R_1(x,t)=u_0\dfrac{\partial u_0}{\partial x}-\nu\dfrac{\partial^2 u_0}{\partial x^2}$
- $R_2(x,t)=u_0\dfrac{\partial u_1}{\partial x}+u_1\dfrac{\partial u_0}{\partial x}-\nu\dfrac{\partial^2 u_1}{\partial x^2},etc$

Step 5: First Few Terms

Let's compute a few terms manually using symbolic expressions:

- $u_0(x,t) = -\sin(\pi x)$
- $\dfrac{\partial u_0}{\partial x} = -\pi\cos(\pi x)$
- $\dfrac{\partial^2 u_0}{\partial x^2} = \pi^2\sin(\pi x)$

So:

$$R_1(x,t) = (-\sin(\pi x))\cdot(-\pi\cos(\pi x)) - \nu\cdot\pi^2\sin(\pi x)$$
$$= \pi\sin(\pi x)\cos(\pi x) - \nu\pi^2\sin(\pi x)$$

Then:

$$u_1(x,t) = h\cdot\int_0^t R_1(x,\tau)d\tau$$
$$= h\cdot t\left[\pi\sin(\pi x)\cos(\pi x) - \nu\pi^2\sin(\pi x)\right]$$

So: $u_1(x,t) = ht\sin(\pi x)\left[\pi\cos(\pi x) - \nu\pi^2\right]$

Final Approximate Solution (Up to First Order)

$$u(x,t) \approx u_0(x,t) + u_1(x,t)$$

Substituting: $u(x,t) \approx -\sin(\pi x) + ht\sin(\pi x)\left[\pi\cos(\pi x) - \nu\pi^2\right]$

Or simplified: $u(x,t) \approx \sin(\pi x)\left[-1 + ht\left(\pi\cos(\pi x) - \nu\pi^2\right)\right]$ (4)

To improve accuracy, you can compute $u_2(x,t), u_3(x,t), \ldots$ similarly. The series converges rapidly for suitable values of hh, typically determined by plotting an "h-curve".

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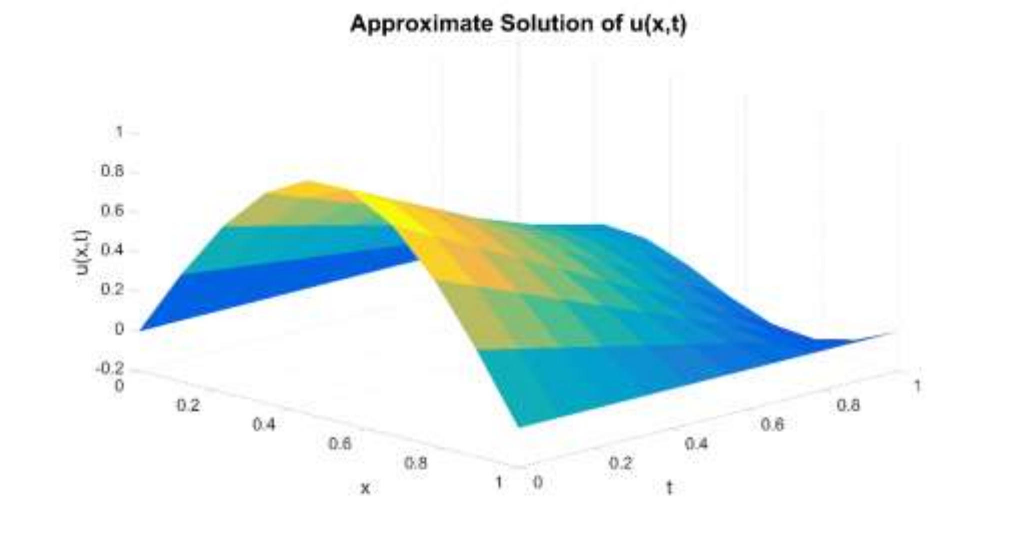

Fig:2 Solution plot of Burger Equation (4)

4. Laplace Adomian Decomposition Method (LADM)

LADM is a hybrid technique that merges the Laplace Transform with ADM to simplify calculations and avoid computing repeated integrals. It is especially useful for time-dependent problems.

Methodology

1. Apply Laplace transform to the given PDE with respect to time t.
2. Transform the PDE into an algebraic equation in the Laplace domain.
3. Apply ADM in the Laplace domain to express the solution as a series.
4. Decompose the nonlinear terms using Adomian polynomials.
5. Use inverse Laplace transform to get the solution in the time domain.

This method often improves convergence and handles initial conditions naturally.

Illustrative Example: Wave Equation

Equation: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

Initial conditions: $u(x,0) = \sin(\pi x), \quad \frac{\partial u}{\partial t}(x,0) = 0$

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Steps:

1. Take Laplace transform in t using:

$$\mathrm{L}\left[\frac{\partial^2 u}{\partial t^2}\right] = s^2\bar{u}(x,s) - su(x,0) - \left.\frac{\partial u}{\partial t}\right|_{t=0}$$

This becomes:

$$s^2\bar{u}(x,s) - s\sin(\pi x) = c^2\frac{\partial^2 \bar{u}}{\partial x^2}$$

2. Rearranged:

$$\frac{\partial^2 \bar{u}}{\partial x^2} - \frac{s^2}{c^2}\bar{u} = -\frac{s}{c^2}\sin(\pi x)$$

3. Use ADM in Laplace space:

Assume: $\bar{u}(x,s) = \sum_{n=0}^{\infty} \bar{u}_n(x,s)$

4. Let: $\bar{u}_0(x,s) = \dfrac{s}{\pi^2 + \dfrac{s^2}{c^2}}\sin(\pi x)$

5. Apply inverse Laplace Transform: $u(x,t) = \sin(\pi x)\cos(c\pi t)$

This is the exact solution, recovered effectively using LADM.

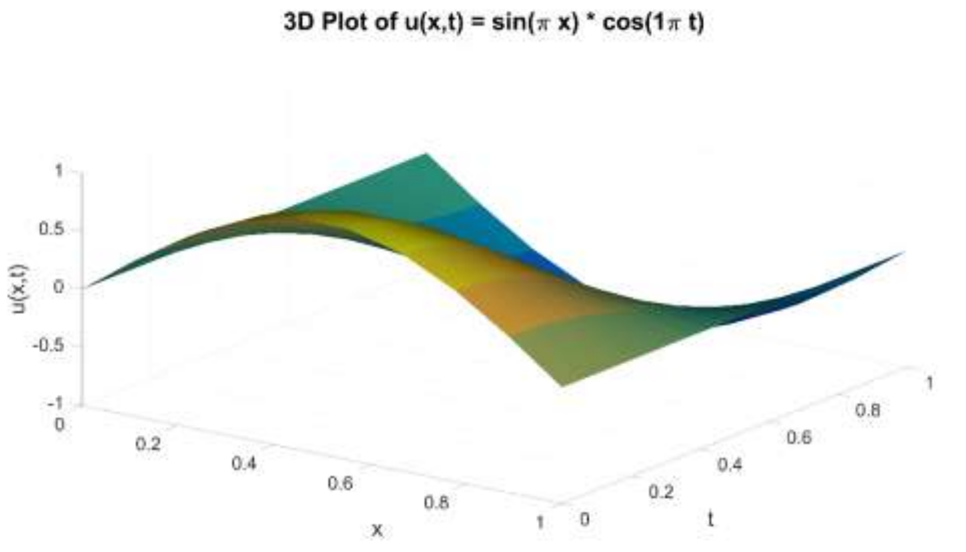

Fig:3 Solution plot of Equation (6) (Solution of Wave Equation (1))

5. Conclusion

The reviewed semi-analytic methods—ADM, HAM, and LADM—are powerful techniques for solving linear and nonlinear PDEs with significant accuracy. ADM is simple and effective for a wide range of problems. HAM offers flexibility and control over convergence, making it ideal for nonlinear problems. LADM combines the advantages of Laplace transforms and decomposition, simplifying computations and enhancing efficiency. Each method contributes uniquely to solving real-world problems in mathematical physics, engineering, and applied sciences.

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