Existence of Solutions of First-Order Nonlinear Differential Equations by Using Fixed Point Theory

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Abstract: This paper investigates the existence of solutions for first-order nonlinear differential equations through the application of fixed-point theory. By transforming the differential equation into an equivalent integral equation, we construct an appropriate operator on a Banach space of continuous functions. Under suitable assumptions on the nonlinear function involved, we employ classical fixed point results such as the Banach Contraction Principle and the Schauder Fixed Point Theorem to establish the existence of at least one solution. This study not only generalizes classical initial value problems but also highlights the applicability of functional analytic methods to nonlinear differential equations.

Keywords: Banach sapce , fixed point , contraction mapping , Existence of solutions, Integral Equations.

1. Introduction

The study of differential equations has its roots in the works of Newton and Leibniz in the 17th century, who introduced calculus to describe natural phenomena mathematically. Over the centuries, mathematicians such as Euler, Bernoulli, and Laplace contributed significantly to solving linear and nonlinear differential equations. A major breakthrough in the rigorous treatment of existence and uniqueness of solutions came through the works of Émile Picard in the late 19th century, who introduced successive approximations (Picard iteration) to prove the existence of solutions to first-order initial value problems. A new perspective emerged in the early 20th century when Stefan Banach established the Fixed Point Theorem in 1922 [1], which provided a powerful and elegant method to prove the existence and uniqueness of solutions by transforming a differential equation into a fixed point problem. Later, Juliusz Schauder generalized this concept to non-contractive operators [2], which allowed proving the existence of solutions even without Lipschitz conditions. These ideas laid the foundation of modern functional analysis and opened the way to solve differential, integral, and integro-differential equations using operator theory.

Classical initial value problems (IVPs) of the form

$$y'(x) = f(x, y(x)), \quad y(x_0) = y_0,$$

are well-understood through the Picard Lindelöf theorem, which ensures the existence and uniqueness of solutions when f satisfies continuity and Lipschitz conditions. However, many physical, biological, and engineering models involve memory effects, hereditary properties, or accumulated influence of past states. Such phenomena are naturally described by equations where the derivative depends not only on y(x) but also on its integral history. This leads to differential equations of the form:

$$y'(x) = f(x, y(x)) + \int_a^x K(x,t,y(t))dt,$$

which are more general than the classical IVP. Traditional methods may fail to handle these nonlocal terms directly. In contrast, fixed point theory provides a flexible framework: by rewriting the equation in integral form and defining an appropriate operator on a Banach space, the problem reduces to finding a fixed point of that operator.

The objective of this paper is to apply fixed point techniques specifically the Banach Contraction Principle and the Schauder Fixed Point Theorem to establish the existence (and under suitable conditions, uniqueness) of solutions of first-order nonlinear differential equations with integral forcing terms. By converting the original differential equation into an equivalent integral operator,

$$(Ty)(x) = y_0 + \int_a^x f(t, y(t))dt + \int_a^x \int_a^t K(t,s,y(s))ds\,dt, \qquad (1)$$

we examine the properties of this operator in the Banach space $C([a,b],\mathbb{R})$.

2. Preliminaries

In this section, we recall some basic definitions and fixed point results that will be used throughout the paper. Most of the concepts are classical and well-documented in standard texts on functional analysis and nonlinear analysis (see [3,5,8]).

2.1 Normed and Banach Spaces

Definition 2.1 : A vector space X over $\mathbb{R}$ (or $\mathbb{C}$) equipped with a function

$\|\cdot\| : X \to \mathbb{R}$ satisfying:

1. $\|x\| \geq 0$ and $\|x\| = 0$ *iff* $x = 0$,
2. $\|\alpha x\| = |\alpha| \cdot \|x\|$ for all scalars α and $x \in X$,
3. $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality),

is called a normed space.

Definition 2.2 : A normed space $(X, \|\cdot\|)$ is called a Banach space if it is complete, i.e., every Cauchy sequence in X converges to an element of X.

In this work, we consider the Banach space $C([a,b],\mathbb{R})$ of continuous real-valued functions on [a,b], equipped with the supremum norm

$$\|y\|_\infty = \sup_{x\in[a,b]} |y(x)|.$$

2.2 Fixed Points:

Definition 2.3: Let $(X, \|\cdot\|)$ be a normed space and $T : X \to X$ a mapping. A point $x \in X$ is called a fixed point of T if T(x) = x.

Fixed point theory is a central tool for proving existence of solutions to differential, integral, and integro-differential equations.

International Scientific Journal of Engineering and Management (ISJEM) ISSN: 2583-6129
Volume: 04 Issue: 10 | Oct – 2025 DOI: 10.55041/ISJEM05138
An International Scholarly || Multidisciplinary || Open Access || Indexing in all major Database & Metadata

Banach Contraction Principle:

Theorem:2.1 (Banach Fixed Point Theorem) [1] Let $(X, \|\cdot\|)$ be a non-empty complete metric (or Banach) space and $T: X \to X$ be a contraction; i.e., there exists a constant 0 < k < 1 such that

$$\| T(x) - T(y) \| \leq k \| x - y \|, \quad \forall x, y \in X. \tag{2}$$

Then T has a unique fixed point in X, and for any $x_0 \in X$, the iterative sequence $x_{n+1} = T(x_n)$ converges to this fixed point.

Theorem:2.2 (Schauder Fixed Point Theorem) [2] Let X be a Banach space, and let $C \subset X$ be a non-empty, closed, bounded, and convex set. If $T: C \to C$ is continuous and compact (i.e., T(C) is relatively compact in X), then T has at least one fixed point in C.

To verify the compactness of the operator in Schauder's theorem, the Arzelà–Ascoli theorem is usually used.

Theorem:2.3 Arzelà–Ascoli Theorem: [9] Let $F \subset C([a,b], \mathbb{R})$. Then F is relatively compact if and only if:

1. F is uniformly bounded, i.e., there exists M > 0 such that $|f(x)| \leq M$ for all $f \in F$ and all $x \in [a,b]$,
2. F is equicontinuous, i.e., for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x_1) - f(x_2)| < \varepsilon$ whenever $|x_1 - x_2| < \delta$ and $f \in F$.

These results allow us to convert a differential equation into a fixed-point problem. In the next section, we will formally state the nonlinear differential equation with integral forcing term and introduce the hypotheses required for our main theorem.

3. Statement of the Problem and Hypotheses

In this work, we consider the following first-order nonlinear differential equation with an integral forcing term:

$$y'(x) = f(x, y(x)) + \int_a^x K(x, t, y(t))dt, \quad x \in [a,b], \tag{3}$$

subject to the initial condition $y(a) = y_0$, where a < b are real numbers, $y_0 \in \mathbb{R}$ is a given constant, and the functions $f:[a,b] \times \mathbb{R} \to \mathbb{R}$, $K:[a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ are assumed to be continuous. Integrating (3) from a to x and using initial condition, we obtain:

$$y(x) = y_0 + \int_a^x f(t, y(t))dt + \int_a^x \left(\int_a^t K(t, s, y(s))ds \right) dt. \tag{4}$$

Thus, we define an operator $T: C([a,b], \mathbb{R}) \to C([a,b], \mathbb{R})$ by

$$(Ty)(x) = y_0 + \int_a^x f(t, y(t))dt + \int_a^x \left(\int_a^t K(t, s, y(s))ds \right) dt. \tag{6}$$

If T has a fixed point in $C([a,b], \mathbb{R})$, i.e., Ty = y, then y(x) is a solution of the original problem (3).

Assumptions (Hypotheses): To apply fixed point theory, we impose the following hypotheses:

- (H1) The function f(x, y) is continuous on $[a,b] \times \mathbb{R}$.
- (H2) The kernel K(x, t, y) is continuous for $(x, t, y) \in [a,b] \times [a,b] \times \mathbb{R}$.
    - (H3) There exist constants $M_1, M_2 > 0$ such that for all $x, t \in [a,b]$ and $y \in \mathbb{R}$,

$$|f(x, y)| \leq M_1(1 + |y|), \quad |K(x, t, y)| \leq M_2(1 + |y|).$$

- (H4) (Lipschitz-type condition) There exist $L_1, L_2 > 0$ such that

$$|f(x,y_1)-f(x,y_2)| \le L_1 |y_1 - y_2|,$$
$$|K(x,t,y_1)-K(x,t,y_2)| \le L_2 |y_1 - y_2|,$$

for all $x,t \in [a,b]$ *and* $y_1, y_2 \in \mathbb{R}$.

Under these hypotheses, our objective is:

- To show that the operator T defined in (5) maps a closed, bounded, convex subset of $C([a,b],\mathbb{R})$ into itself.
- To prove that T has a fixed point by using either:
  - Banach Contraction Principle (if the contraction condition holds), or
  - Schauder Fixed Point Theorem (if T is compact and continuous).
- To conclude that a fixed point of T is a solution of (3)

This establishes the existence of at least one solution to the given first-order nonlinear differential equation with integral forcing term.

4. Main Existence Theorem

In this section, we prove the existence of at least one solution to the nonlinear differential equation with integral forcing term given in (3), using the operator T defined in (5). The proof is based on the Schauder Fixed Point Theorem and, under stronger conditions, on the Banach Contraction Principle.

Theorem 4.1: Suppose that the hypotheses (H1)--(H4)} from Section 3 are satisfied. Then:

1. There exists at least one function $y \in C([a,b],\mathbb{R})$ such that Ty = y, i.e., y is a solution of the differential equation (3) with initial condition.
2. Moreover, if $L_1(b-a) + \frac{L_2(b-a)^2}{2} < 1,$

then the operator T$is a contraction, and the solution is unique.

Proof: Define the closed, bounded, convex subset

$$B_R = \left\{ y \in C([a,b],\mathbb{R}) : \| y \|_\infty \le R \right\},$$

where R > 0 will be chosen sufficiently large.

Step 1: T maps BR into itself: For any $y \in B_R$ *and* $x \in [a,b]$,

$$|(Ty)(x)| \le |y_0| + \int_a^x |f(t,y(t))|\,dt + \int_a^x \int_a^t |K(t,s,y(s))|\,ds\,dt.$$

Using (H3), we get:

$$|(Ty)(x)| \le |y_0| + M_1(b-a)(1+R) + M_2 \frac{(b-a)^2}{2}(1+R).$$

Choose R such that: $R \ge |y_0| + M_1(b-a)(1+R) + M_2 \frac{(b-a)^2}{2}(1+R).$

This is possible since the right-hand side grows linearly with R. Hence $T(B_R) \subset B_R$.

Step 2: T is continuous: Let $y_n \to y$ in C([a,b]). Then, using continuity of f and K from (H1)--(H2) and the Dominated Convergence Theorem,

$Ty_n(x) \to Ty(x)$ uniformly on $[a,b]$. Thus, T is continuous.

Step 3: T is compact: Let $F = T(B_R)$. From Step 1, F is uniformly bounded. For $x_1, x_2 \in [a,b]$

$$|(Ty)(x_1)-(Ty)(x_2)| \le \int_{x_1}^{x_2} |f(t,y(t))|\,dt + \int_{x_1}^{x_2}\int_a^t |K(t,s,y(s))|\,ds\,dt.$$

Using (H3), this difference can be made arbitrarily small if $|x_1 - x_2|$ is small. Hence, F is equicontinuous. By the Arzelà–Ascoli Theorem, T(BR) is relatively compact.

Step 4: Application of Schauder Fixed Point Theorem: Since T is continuous, compact, and maps the closed, bounded, convex set BR into itself, the Schauder Fixed Point Theorem guarantees that T has a fixed point in BR. Hence, there exists $y \in C([a,b],\square)$ such that Ty = y.

Step 5: Uniqueness under contraction condition: We now assume

$L_1(b-a) + \frac{L_2(b-a)^2}{2} < 1$. For any $y_1, y_2 \in C([a,b])$,

$$|(Ty_1)(x)-(Ty_2)(x)| \le \int_a^x L_1 |y_1(t)-y_2(t)|\,dt + \int_a^x\int_a^t L_2 |y_1(s)-y_2(s)|\,ds\,dt.$$

Thus,

$$\| Ty_1 - Ty_2 \|_\infty \le \left(L_1(b-a) + \frac{L_2(b-a)^2}{2} \right) \| y_1 - y_2 \|_\infty .$$

Hence, T is a contraction and by the Banach Contraction Principle, T has a unique fixed point. This completes the proof.

5. Verified Example

In this section, we present a concrete example to demonstrate the applicability of Theorem 4.1. We consider a specific nonlinear differential equation with an integral forcing term and verify that all assumptions (H1)--(H4) are satisfied.

Example: 1 Consider the differential equation:

$$y'(x) = \frac{1}{2}y(x) + \int_0^x \frac{1}{4}e^{-(x-t)}y(t)\,dt, \quad x \in [0,1], \tag{7}$$

with the initial condition

$$y(0) = 1. \tag{8}$$

Here,

$f(x,y) = \frac{1}{2}y, \quad K(x,t,y) = \frac{1}{4}e^{-(x-t)}y(t),$ where a=0, b=1, and y_0= 1.

International Scientific Journal of Engineering and Management (ISJEM) ISSN: 2583-6129
Volume: 04 Issue: 10 | Oct – 2025 DOI: 10.55041/ISJEM05138
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Verification of Assumptions:

- [(H1)] **Continuity of f(x,y):** $f(x,y)=\frac{1}{2}y$ is continuous in both variables.
- [(H2)] **Continuity of K(x,t,y):** $K(x,t,y)=\frac{1}{4}e^{-(x-t)}y$ is continuous in (x,t,y) on $[0,1]\times[0,1]\times\mathbb{R}$.
- [(H3)] **Growth condition:** For all $x,t\in[0,1]$ and $y\in\mathbb{R}$,

$$|f(x,y)|=\frac{1}{2}|y|\le\frac{1}{2}(1+|y|),$$
$$|K(x,t,y)|=\frac{1}{4}e^{-(x-t)}|y|\le\frac{1}{4}(1+|y|).$$

Hence, (H3) holds with $M_1 = \frac{1}{2}$ and $M_2 = \frac{1}{4}$.

- [(H4)] **Lipschitz condition:** $|f(x,y_1)-f(x,y_2)|=\frac{1}{2}|y_1-y_2|\Rightarrow L_1=\frac{1}{2}.$

$$|K(x,t,y_1)-K(x,t,y_2)|=\frac{1}{4}e^{-(x-t)}|y_1-y_2|\Rightarrow L_2=\frac{1}{4}.$$

Contraction Condition: We compute:

$$L_1(b-a)+\frac{L_2(b-a)^2}{2}=\frac{1}{2}\cdot1+\frac{1}{4}\cdot\frac{1^2}{2}=\frac{1}{2}+\frac{1}{8}=\frac{5}{8}<1.$$

Hence, by Theorem 4.1 , the operator T is a contraction on C([0,1]). Therefore:

- A unique solution y(x) of Example 1 exists on [0,1].
- The iterative sequence defined by $y_{n+1}=Ty_n$ converges to this unique solution.

Approximate Solution (Optional Illustration): Starting with $y_0(x)=1$, we compute:

$$y_1(x)=1+\int_0^x\frac{1}{2}\cdot1dt+\int_0^x\int_0^t\frac{1}{4}e^{-(t-s)}\cdot1dsdt=1+\frac{x}{2}+\frac{1}{4}\left(x-1+e^{-x}\right).$$

This sequence converges uniformly to the true solution of \eqref{example_eq}.

6. Conclusion

In this paper, we have investigated the existence of solutions of first-order nonlinear differential equations with an integral forcing term using fixed point theory. By converting the differential equation into an equivalent integral equation, we defined an operator on the Banach space $C([a,b],\mathbb{R})$. Under appropriate assumptions on the nonlinear functions f(x,y) and K(x,t,y), we proved the existence of solutions using the Schauder Fixed Point Theorem. Furthermore, by imposing a contraction condition, we established the uniqueness of the solution via the Banach Contraction Principle. A verified example was presented to illustrate the validity of our theoretical results. The techniques used in this work can be extended further to fractional differential equations, delay differential equations, and integro-differential equations. Future research may involve studying stability, continuous dependence on initial data, and numerical approximations of these solutions.

ISSN: 2583-6129
DOI: 10.55041/ISJEM05138

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