

Metric Topology in Airline Network Optimization: A Study of Distance Functions, Connectivity, and Delay Propagation

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ABSTRACT

This paper explores the application of metric topology in airline network optimization. A metric space is defined to represent distances between airports, incorporating factors such as travel time, cost, and delay. Using these metrics, concepts of open and closed sets are employed to analyse connectivity, route structure, and accessibility within airline networks.

The study also examines delay propagation through metric-based models, highlighting its impact on network efficiency and robustness. By linking topological concepts with real-world airline systems, this work demonstrates how metric topology can support improved route planning, scheduling, and disruption management.

KEYWORDS

Metric Space, Metric Topology, Airline Networks, Distance Functions, Network Connectivity, Open Sets, Closed Sets, Route Optimization, Delay Propagation, Transportation Systems

1. INTRODUCTION

Topology is a fundamental branch of mathematics concerned with the study of spaces and the relationships between their elements under continuous transformations. Within this framework, metric spaces play a central role by introducing a well-defined notion of distance between elements of a set. This concept allows abstract spaces to be analysed using distance functions, forming a bridge between geometry and topology.

A metric space naturally induces a topological structure, where concepts such as open sets, closed sets, convergence, and continuity are defined in terms of distance. These ideas provide a rigorous foundation for analysing connectivity and structure in complex systems.

In the context of airline transportation networks, metric spaces offer a powerful tool to model relationships between airports by incorporating not only geographical distance but also operational factors such as travel time, cost, and delay. This enables a more realistic representation of network behaviour. Topological concepts derived from these metrics can then be used to study route efficiency, network connectivity, and the propagation of delays across interconnected flights.

Thus, the integration of metric topology with airline network analysis provides both a theoretical foundation and practical insights for improving route planning, scheduling, and system robustness.

2. IMPORTANT DEFINITIONS (AIRLINE-FOCUSED)

2.1 Topological Space

Topology studies the structure of sets based on relationships between their elements rather than exact numerical values. A topological space is defined as a pair (X, τ) where X is a non-empty set and τ is a collection of subsets of X satisfying specific axioms: inclusion of the empty set and the whole set, closure under arbitrary unions, and closure under finite intersections.

In the context of airline systems, the set X can be interpreted as the collection of all airports in a network. The subsets in τ represent groups of airports that share a common operational property, such as being reachable within a certain time or cost threshold. These subsets can be viewed as **regions of connectivity**.

Unlike traditional geometric models, topology allows the study of airline networks without relying strictly on numerical distance. Instead, it focuses on how airports are connected through routes. This is useful in identifying clusters of airports, such as hub-based systems, where multiple destinations are connected through a central airport.

For example, a set of airports that can be reached within two hours from a major hub forms an open set. Similarly, airports that are strongly interconnected through frequent flights can be grouped into a topological neighbourhood. This abstraction helps analyse network structure, resilience, and connectivity patterns without being restricted to physical distance alone.

2.2 Metric and Metric Space

A metric introduces a quantitative way to measure distance between elements of a set. Formally, a metric is a function $d : X \times X \rightarrow \mathbb{R}$ that satisfies non-negativity, symmetry, and the triangle inequality.

In airline networks, the concept of distance is more complex than simple geographical separation. The metric can be defined in multiple ways depending on the objective of the analysis. For instance, distance between two airports may represent:

- Flight duration
- Ticket cost
- Number of layovers
- Probability of delay

Thus, the airline network can be modelled as a **metric space (X, d)** where X represents airports and $d(x, y)$ represents an operational distance between them.

This flexibility allows the same mathematical structure to be used for different optimization goals. For example, minimizing travel time leads to one type of metric, while minimizing cost or delay leads to another.

Consider two examples:

- The distance between Chennai and Delhi may be defined as the flight time, making it a time-based metric.
- Alternatively, the distance may include a weighted combination of cost and delay, providing a more realistic measure for decision-making.

Examples of Metrics in Airline Context

1. Discrete Metric

The discrete metric distinguishes only whether two airports are identical or not. It is useful in simplified models where only connectivity is considered.

- Example 1: Checking whether two flight nodes represent the same airport
- Example 2: Binary models of network connectivity

2. Euclidean Metric

The Euclidean metric represents straight-line (great-circle) distance between airports.

- Example 1: Determining shortest possible air route
- Example 2: Estimating fuel consumption and flight range

3. Taxicab Metric

The taxicab metric measures distance as the sum of component-wise differences. While it is more suitable for road networks, it can model indirect airline routes.

- Example 1: Multi-leg journeys (Hyderabad → Mumbai → Delhi)
- Example 2: Routes constrained by air traffic corridors

4. Supremum Metric

This metric considers the maximum deviation among components.

- Example 1: Maximum delay in a connecting itinerary
- Example 2: Worst-case travel time across route segments

2.3 Open Ball and Closed Ball

An open ball in a metric space represents all points within a certain distance from a centre point. In airline terms, this can be interpreted as all airports that are reachable within a specified limit.

An open ball excludes boundary cases, while a closed ball includes them.

- Example 1: All airports reachable from Bangalore within 2 hours form an open ball
- Example 2: All routes costing less than or equal to a fixed budget form a closed ball

These concepts are useful in defining **reachable regions** in airline networks.

2.4 Open Sets

A set is open if every point in it has a surrounding neighbourhood entirely contained within the set. In airline systems, open sets represent regions where connectivity is stable and flexible.

- Example 1: Airports with frequent and reliable connections form an open set
- Example 2: Regions with minimal delay variation can be treated as open sets

Open sets help in analysing accessibility and smooth connectivity across the network.

2.5 Closed Sets

A closed set contains all its boundary points. In airline networks, closed sets represent regions with clearly defined operational limits.

- Example 1: Airports within a fixed regulatory or operational boundary
- Example 2: A complete set of routes under a specific airline network

Closed sets are useful for defining **complete and bounded systems**.

2.6 Properties of Metric Topology

Metric topology provides several important structural properties:

- Open balls form a basis for defining the topology
- Every singleton set is closed
- Metric spaces satisfy separation properties, ensuring distinct points can be separated
- Closure of a set includes all points arbitrarily close to it

In airline systems, these properties help model how local changes (like delays) affect nearby routes and how clusters of airports behave as connected components.

3. METRIC TOPOLOGY AND AIRLINE NETWORK STRUCTURE

Metric topology connects the abstract definitions of metric spaces with the structural properties of real-world systems. In the context of airline networks, this framework helps translate distance-based relationships into meaningful insights about connectivity, clustering, and operational behaviour.

A metric defined on a set of airports induces a topology in which open sets represent regions of strong connectivity. These regions can be interpreted as clusters of airports that are efficiently reachable from one another under a chosen distance function, such as travel time or cost.

Key Interpretations in Airline Networks:

- **Topology from Metric:** Every metric generates a topology where open balls define neighbourhoods. In airline systems, this corresponds to airports that are reachable within a certain threshold.
- **Connectivity Structure:** Topology helps identify whether the network is well-connected or fragmented.
- **Neighbourhood Systems:** Local neighbourhoods represent reachable zones from a given airport.

Two Concrete Examples:

Example 1 (Hub-Based Connectivity):

If Dubai acts as a hub, the set of airports reachable within 3 hours forms a neighbourhood. This shows how hubs create dense topological regions.

Example 2 (Delay Spread as Topology):

If delays propagate from one airport to nearby connected airports, the affected region behaves like a topological closure of that point.

4. IMPORTANT LEMMAS (INTERPRETATION)

Lemma 4.1: Basis of Open Balls

Statement: Let (X, d) be a metric space. Then the collection of all open balls

$$\{B(x, r) : x \in X, r > 0\}$$

forms a basis for a topology on X .

Explanation: Every open set in a metric space can be expressed as a union of open balls. Hence, open balls generate the topology induced by the metric.

Lemma 4.2: Open Balls in \mathbb{R}

Statement: In the metric space (\mathbb{R}, d) where $d(x, y) = |x - y|$, every open ball $B(x, r)$ is equal to the open interval $(x - r, x + r)$

Explanation: This establishes that the standard topology on \mathbb{R} is induced by the usual metric.

Lemma 4.3: Discrete Metric and Discrete Topology

Statement: Let d be the discrete metric on a set X . Then every subset of X is open, and the induced topology is the discrete topology.

Explanation: In a discrete metric space, each singleton set is open, which implies that all subsets are open.

Lemma 4.4: Convergence in Metric Spaces

Statement: A sequence (x_n) in a metric space (X, d) converges to a point $x \in X$ if and only if

$$\lim_{n \rightarrow \infty} d(x_n, x) = 0$$

Explanation:

Convergence is characterized entirely in terms of the metric, meaning that the distance between sequence elements and the limit approaches zero.

Lemma 4.5: Closure in Metric Spaces

Statement: The closure of a subset $A \subseteq X$ in a metric space (X, d) consists of all points $x \in X$ such that

$$d(x, A) = 0$$

Explanation:

A point belongs to the closure of a set if its distance from the set can be made arbitrarily small.

Lemma 4.6: Hausdorff Property

Statement: Every metric space is Hausdorff. That is, for any two distinct points $x, y \in X$, there exist disjoint open sets U and V such that

$$x \in U, y \in V$$

Explanation: This ensures that distinct points can always be separated by neighbourhoods.

Lemma 4.7: Isometric Metric Spaces

Statement: Two metric spaces (X, d) and (Y, e) are said to be isometric if there exists a bijective function $f: X \rightarrow Y$ such that

$$d(x_1, x_2) = e(f(x_1), f(x_2)) \quad \forall x_1, x_2 \in X$$

An isometry preserves distances exactly, implying that the two spaces have identical metric structure.

5. APPLICATION OF METRIC IN AIR TRANSPORT

Metric spaces provide a structured way to represent airline systems, where each airport is treated as a point and the distance between two airports is defined using a suitable function. This distance does not have to be purely geographical; it can represent travel time, ticket cost, or operational delays.

One major application is in **route selection**. Airlines do not always choose the shortest physical path. Instead, they use distance measures that consider time efficiency, fuel usage, and scheduling constraints to identify better routes.

Metrics are also useful in analysing **connecting flights**. When passengers travel through multiple airports, the total distance depends on each segment of the journey. The triangle inequality ensures that the total journey remains logically consistent and helps compare direct and indirect routes.

Another important use is in **delay analysis**. Since flights are interconnected, a delay at one airport can influence others. By defining distance in terms of connectivity or dependency, it becomes possible to study how disruptions spread across the network.

Metric concepts also help in identifying **important airports (hubs)**. Airports that are effectively “closer” to many others, in terms of time or connectivity, play a central role in maintaining efficient operations.

6. METRIC LEARNING (CONTROLLED, RELEVANT)

Let X be the set of airports. A learned metric $d(x, y)$ is designed by combining multiple operational factors such as travel time, ticket cost, and delay. Instead of treating these factors independently, they are integrated into a single distance function that satisfies the properties of a metric.

A general form of such a metric can be defined as:

$$d(x, y) = w_1T(x, y) + w_2C(x, y) + w_3D(x, y)$$

where:

- $T(x, y)$ represents travel time
- $C(x, y)$ represents cost
- $D(x, y)$ represents delay impact
- w_1, w_2, w_3 are non-negative weights

This construction ensures that the distance function remains non-negative and symmetric. With appropriate choice of weights, the triangle inequality can also be maintained, making it a valid metric.

7. CONCLUSION

This study highlighted how metric topology provides a meaningful framework for understanding the structure and behaviour of airline networks. By defining appropriate distance functions, airline systems can be viewed as topological spaces where connectivity, clustering, and accessibility are naturally described through open sets and neighbourhoods.

A key insight of this work is the role of topology in analysing **delay propagation**. Since airline networks are highly interconnected, disruptions at one airport can influence others through closely linked routes. When distance is defined in terms of operational dependence rather than geography, airports that are strongly connected become “closer” in the metric space, making it possible to identify regions where delays are likely to spread rapidly.

This topological perspective allows delays to be interpreted not as isolated events but as phenomena that evolve over connected structures. As a result, metric-based models can support better identification of critical hubs, improve scheduling strategies, and enhance the overall resilience of airline networks.

In summary, the combination of metric spaces and topology offers a mathematically consistent and practically relevant approach for studying airline connectivity and managing delay propagation in complex transportation systems.

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