Some Fixed Point Results on Compact Metric Spaces

Swati Ankush Chavan[1], Varsha D. Borgaonkar[2]
[1]*P.G. Department of Mathematics, N.E.S. Science College, Nanded, India - 431 602*
[2]*P.G. Department of Mathematics, N.E.S. Science College, Nanded, India - 431 602*

Abstract: This paper aims to study some generalised fixed point results in a compact metric space. It mainly focuses on the existence and uniqueness of fixed point of self-mappings on a metric space and its generalizations. This paper uses iterative techniques to show the existence of a unique fixed point for a self-mapping satisfying certain generalized contractive conditions.
Keywords: Compact Space, Metric Space Contraction, Fixed Point, Convergence sequence.

1. INTRODUCTION

Fixed point theory is a fascinating subject with many applications in various fields of Mathematics, such as differential equations and numerical analysis. Also, the existence of a Nash equilibrium in game theory can be formulated as a fixed-point problem. Fixed Point Theory has an important role in Mathematical economics. It is interesting to discuss the existence and uniqueness of fixed points for the self-mappings defined on compact metric spaces. There are several generalizations of the classical contraction mapping theorem of Banach. In 1961, Edelstein [4] established the existence of a unique fixed point of a self-map T of a compact metric space satisfying the inequality $d(Tx, Ty) < d(x, y)$, which is a generalization of the Banach Contraction Principle. Fisher [5], has proved some results on compact metric spaces. Here we are found some fixed point results for contractive mappings on a compact metric Space.

2. BASIC DEFINITION & EXAMPLES

Definition 2.1: (Metric Space): Let X be a non-empty set. A metric on X is a real-valued function $d: X \times X \to R$ which satisfies the following conditions:

1. $d(x,y) \geq 0,$ $\quad \forall\, x,y \in X,$
2. $d(x,y) = 0$ if and only if $x = y$ $\quad \forall\, x,y \in X,$
3. $d(x,y) = d(y,x),$ $\quad \forall\, x,y \in X$ (Symmetry),
4. $d(x,y) \leq d(x,z) + d(z,y),$ $\quad \forall x,y,z \in X$ (Triangle inequality)

Definition 2.2: (Open Set): A subset G of a metric space (X,d) is said to be open in X, with respect to the metric d, if G is a neighbourhood of each of its points. In other words, if for each $a \in G$, there is an $r > 0$, such that, $s_r(a) \subseteq G$.

Example 2.1: On the real line with the usual metric, the singleton set is open.

Definitions 2.3: (Open Cover): Let (X,d) be a metric space. A family of subsets $\{A_\propto\}$ in X is called a cover of any subset A of X if $A \subseteq \bigcup_{\propto\in\wedge} A_\propto$, $\wedge$ is any non-empty index set. If each $A_\propto$, $\propto\in\wedge$, is an open set in X, then the cover $\{A_\propto\}$ is called an open cover of A.

Definition 2.4: (Open Subcover): A subfamily of the family $\{A_\propto\}$ which itself is an open cover, is called an open subcover of A.

International Scientific Journal of Engineering and Management (ISJEM) ISSN: 2583-6129
Volume: 04 Issue: 04 | April – 2025 DOI: 10.55041/ISJEM02606
An International Scholarly || Multidisciplinary || Open Access || Indexing in all major Database & Metadata

Definition 2.5: (Finite Subcover): If the number of members in the subfamily is finite, it is called a finite subcover of A.

Definition 2.6: (Compact Metric Space): A subset A of a metric space (X, d) is said to be compact if every open cover of A admits of a finite subcover, i.e., for each family of open subsets $\{G_\propto\}$ of X, for which $A \subseteq \bigcup_{\propto \in \wedge} G_\propto$, there exists a finite subfamily say $\{G_{\propto 1}, G_{\propto 2}, \dots, G_{\propto n}\}$ such that $A \subseteq \bigcup_{i=1}^{n} G_{\propto i}$.

Example 2.2: Any closed interval with the usual metric is compact.

Definition 2.7: Let (X, d) be a metric space. A mapping $T: X \to X$ is called

i. Contraction: If $d(Tx, Ty) \leq \alpha d(x, y)$, $\forall\, x, y \in X$, $\alpha \in [0,1)$.
ii. Contractive: If $d(Tx, Ty) < f(x, y)$, $\forall\, x, y \in X$ with $x \neq y$.

Theorem 2.1: (Banach Contraction Theorem): A contraction T on a complete metric space (X, d) has a unique fixed point. If z is the fixed point of the mapping T, then for any $x \in X$, the sequence $\{T^n x\}$ of iterates converges to z.

Definition 2.8: (Fixed point): Let (X, d) be a metric space, then a point is said to be a fixed point of the self-map $f : X \to X$ if $f(x) = x$.

Example2.3: Let $f: R \to R$ defined as, $f(x) = x^2\ \forall\ x \in R$, then $x = 0$ and $x = 1$, are the fixed points of the mapping f.

3. MAIN RESULTS

Theorem 3.1: Let T be a continuous self-map of a compact metric space satisfying the conditions,

$$d(T(x), Ty) < \alpha.[d(x, T(x)) + d(x, T(y))] + \beta.[d(y, T(x)) + d(y, T(y))] \quad \text{...(3.1)}$$

where, $0 < \alpha < 1$ and $0 < \beta < 1$ are such that $\alpha + \beta < \frac{1}{3}$. Then T has a unique fixed point in X.

Proof: We define $f : X \to [0, \infty]$ as, $f(x) = d(x, T(x))$ $\forall x \in X$.

Case I: If $x = T(x)$ for some $x \in X$ then, x willl be a fixed point of X.

Case II: Suppose $x \in X$ is such that $T(x) \neq x$.

$$\therefore \quad f(T(x)) = f\,(T(x), T^2(x)).$$
$$f(T(x)) = f\,(T(x), T(y)).$$

By applying (3.1), we get,

$$f(T(x)) < \alpha\,[d(x, T(x)) + d(x, T^2(x))] + \beta[d(T(x), T(x)) + d(T(x), T^2(x))]$$
$$f(T(x)) < \alpha[d(x, T(x)) + d\,(x, T(x)) + d(T(x), T^2(x))] + \beta[d(T(x), T^2(x)) + d(T(x), T^2(x))]$$
$$f(T(x)) < \alpha\big[f(x) + f(x) + f\big(T(x)\big)\big] + \beta\, f\big(T(x)\big)$$
$$< 2\alpha\, f(x) + (\alpha + \beta)\, f\big(T(x)\big). \quad \dots(3.2)$$

If we have, $f(x) \leq f\big(T(x)\big)$, for $x \neq T(x)$, then by (3.2) we have

$$f\big(T(x)\big) < (3\alpha + \beta)\, f\big(T(x)\big)$$

This is not possible, since $(3\alpha + \beta) < 1$.

Therefore, we are left with only the possibility that

$$f(T(x)) < f(x) \qquad \text{for } x \neq T(x) \quad \dots(3.3)$$

As T and d are continuous, f is also a continuous function on a compact metric space X. Hence, by the theorem, f attains its minimum on X. Suppose, f attains a minimum at $x_0 \in X$.

Let, $f(x_0) = \min\,\{f(x) : x \in X\}$ $\quad \dots(3.4)$

Now, we show that x_0 is a fixed point of f. If it is not, we have, $x_0 \neq T(x_0)$ hence, (3.3) gives, $f(T(x_0)) < f(x_0)$. This contradicts (3.4), thus, we have, $x_0 = T(x_0)$.

x_0 is fixed point of T.

Uniqueness: Suppose T has two fixed points, say $x_{0,}\ y_0$ in X. Therefore, we have, $T(x_0) = x_0$ and $T(y_0) = y_{0.}$

Consider,

$$d(x_0, y_0) = d(T(x_0), T(y_0))$$
$$< \alpha\,[d(x_0, T(x_0)) + d(x_0, T(y_0))] + \beta[(y_o, T(y_0)) + d(y_0, T(x_0))]$$
$$< \alpha\,[d(x_0, x_0) + d(x_0, y_0)] + \beta[(y_o, y_0) + d(y_0, x_0)]$$
$$d(x_0, y_0) < \alpha[0 + d(x_0, y_0)] + \beta[d(y_0, x_0) + 0]$$
$$< (\alpha + \beta)\, d(x_0, y_0)$$

This is not possible, since $\alpha + \beta < 1$. Therefore, T has a unique fixed point in X.

Theorem 3.2: Let T be a self-map on a compact metric space (X, d) and then T satisfies (3.1).Then the sequence $\{T^n(x)\}$ converges to the unique fixed point of T.

Proof: Let x_0 be a unique fixed point of T, we have $T(x_0) = x_0$. Now, we define,

$$d_n = d(T^n(x), x_0) \qquad \forall x \in X, x \neq x_0.$$

Case I: If $d_n = 0$ for some $n = N$,

Therefore, we have, $T^m(x) = x_0 \qquad \forall\, m \geq n$ ... (3.5)

Hence, $\{T^n(x)\}$ converges to x_0.

Case II: If $d_n \neq 0 \qquad \forall\, n$, then, we have,

$$d_{n+1} = d(T^{n+1}(x), x_0)$$
$$= d(T^{n+1}(x), T^{n+1}(x_0))$$
$$= d(T(T^n(x), T(T^n(x_0))$$
$$< \alpha[d(T^n(x), T(T^n(x)) + d(T^n(x), T^{n+1}(x_0))]$$
$$+\beta[d(T^n(x_0), T(T^n(x)) + d(T^n(x_0), T(T^n x_0))$$
$$< \alpha[d(T^n(x), T^{n+1}(x)) + d(T^n x, T^{n+1} x_0)] + \beta[d(T^n x_0, T^{n+1}(x)) + d(T^n(x_0), T^{n+1}(x_0))]$$
$$d_{n+1} < \alpha\big[d(T^n(x), x_0) + d(x_0, T^{n+1}(x)) + d(T^n(x), x_0)\big] + \beta[d(T^n(x_0), x_0) + d(x_0, T^{n+1}(x) + d(T^n(x_0), T^{n+1}(x_0))]$$

But, x_0 is fixed point of T, therefore, we have, $T^n(x_0) = T^{n+1}(x_0) = x_0$, which gives,

$$d_{n+1} < \alpha[d(T^n(x), x_0) + d(T^{n+1}(x)(x_0)) + d(T^n(x), x_0)] + \beta[d(x_0, x_0) + d(T^{n+1}(x), x_0) + d(x_0, x_0)]$$
$$d_{n+1} < \alpha[d_n + d_{n+1} + d_n] + \beta[0 + d_{n+1} + 0]$$
$$< 2\alpha d_n + (\alpha + \beta) d_{n+1}$$
$$[1 - (\alpha + \beta)]d_{n+1} < 2\alpha d_n$$
$$d_{n+1} < \left[\frac{2\alpha}{1 - (\alpha + \beta)}\right] d_n$$
$$d_{n+1} < d_n \qquad \text{since, } \alpha + \beta < \tfrac{1}{3}.$$

$\therefore \{d_n\}$ is strictly decreasing sequence of positive real numbers and hence, it is a convergent sequence. Suppose, $\lim_{n\to\infty} d_n = r\ (say)$ $(r \geq 0)$, where, $r = \inf\, \{d_n | n \,\epsilon\, N\}$

Now, $\lim_{n\to\infty} d_n = r \quad \Rightarrow \quad \lim_{n\to\infty} d(T^n(x), (x_0)) = r$

As, X is compact $\{T^n(x)\}$ has a convergent subsequence $\{T^{n_k}(x)\}$, which converges to $z \in X$.

$$\therefore \lim_{k\to\infty} T^{n_k}(x) = z \quad , \qquad z \in X$$

As, d is a continuous function, we have,

International Scientific Journal of Engineering and Management (ISJEM) ISSN: 2583-6129
Volume: 04 Issue: 04 | April – 2025 DOI: 10.55041/ISJEM02606
An International Scholarly || Multidisciplinary || Open Access || Indexing in all major Database & Metadata

$$\therefore \lim_{k\to\infty} d(T^{n_k}(x),(x_0)) = d(z,x_0)$$

$i.e.,$ $\lim_{k\to\infty} d_{n_k} = d(z,x_0)$ ... (3.6)

But $\{d_{n_k}\}$ is a subsequence of $\{d_n\}$, therefore, we have,

$$\lim_{k\to\infty} d_{n_k} = r \qquad \text{... (3.7)}$$

Hence, (3.6) and (3.7) give,

$$d(z,x_0) = r \qquad \text{... (3.8)}$$

Moreover,

$$\lim_{k\to\infty} T^{n_k}(x) = z \Rightarrow \lim_{k\to\infty} T^{n_k+1}(x) = \lim_{k\to\infty} T(T^{n_k}(x))$$

$$\lim_{k\to\infty} T^{n_k}(x) = z \Rightarrow \lim_{k\to\infty} T^{n_k+1}(x) = \lim_{k\to\infty} T\big(T^{n_k}(x)\big) = T(z)$$

Since, T is a continuous mapping.
Hence,

$$\lim_{k\to\infty} d_{n_k+1} = \lim_{k\to\infty} d\left(T^{n_k+1}(x),(x_0)\right) = d(T(z),x_0)$$

$\therefore$ We get,

$$\lim_{k\to\infty} d_{n_k+1} = d(Tz,x_0) \qquad \text{... (3.9)}$$

As $\{d_{n_k+1}\}$ is a subsequence of $\{d_n\}$, we have,

$$\lim_{k\to\infty} d_{n_k+1} = r \qquad \text{... (3.10)}$$

Now, (3.9) and (3.10) gives,

$$d(T(z),x_0) = r \qquad \text{... (3.11)}$$

Again, (3.8) and (3.11) we have,

$$d(T_z,x_0) = d(z,x_0) = r \qquad \text{... (3.12)}$$

Now, we will prove that $r = 0$.
Suppose, $r \neq 0$.
Consider, for $T(z) \neq x_0$, we have,

$d(T(z),x_0) = d\big(T(z),T(x_{0)}\big) \qquad \because Tx_0 = x_0$

$d(T(z),x_0) < \alpha[d(T(z),x_0) + d(z,T(x_0))] + \beta[d(x_0,T(z)) + d(x_0,T(x_0))]$

$d(T(z),x_0) < \alpha[d(T(z),z) + d(x_0,z) + d(z,x_0)] + \beta[d(x_0,T(z)) + d(x_0,x_0)]$

$d(T(z),x_0) < \alpha[d(T(z),x_0) + d(x_0,z) + d(x_0,z)] + \beta[d(T(z),x_0)]$

$< (\alpha+\beta)d(T(z),x_0) + 2\alpha\, d(x_0,z)$

$(1-(\alpha+\beta)]\, d(T(z),x_0) < 2\alpha\, d(x_0,z)$

$$d(T(z),x_0) < \left[\frac{2\alpha}{1-(\alpha+\beta)}\right] d(x_0,z)$$

$d(T(z),x_0) < d(x_0,z) \qquad \because \alpha+\beta < \frac{1}{3}.$

This is a contradiction to (3.12). Hence, we have $r = 0$.
$\therefore$ By (3.12), we have,

$$d(T(z),x_0) = d(z,x_0)$$

Hence, the sequence $\{d_n\}$ converges to $r = 0$ as $n \to \infty$

$i.e.$ $\lim_{n\to\infty} d_n = 0 \Rightarrow \lim_{n\to\infty} d(T^n(x),x_0) = 0$

Hence, the sequence $\{T^n(x)\}$ converges to the fixed point x_0 of T. Moreover, $\{T^n(x)\}$ converges to a unique fixed point x_0 of T.

4. APPLICATIONS:

Let, $X = [0,1]$ be a compact metric space with respect to the usual metric d defined as, $d(x, y) = |x - y| \quad \forall\, x, y \in X$. Define, $T: X \to X$ as, $T(x) = \frac{x}{4} \quad \forall\, x \in X$.

Choose $\alpha = \frac{1}{4}$, $\beta = \frac{1}{24}$, Such that, $\alpha + \beta = \frac{1}{4} + \frac{1}{24} < \frac{1}{3}$

Moreover, we have, for $x \neq y$

$$\begin{aligned}
&\alpha[d(x, Tx) + d(x, Ty)] \\
&= \frac{1}{4}\left[\left|x - \frac{x}{4}\right| + \left|x - \frac{y}{4}\right|\right] \\
&= \frac{1}{4}\left[\left|\frac{3x}{4}\right| + \left|\frac{4x-y}{4}\right|\right] \\
&\geq \frac{1}{4}\left|\frac{4x-y}{4}\right| \\
&\geq \frac{1}{4}\left|\frac{4x-4y}{4}\right| \\
&\geq \frac{1}{4}|x - y| \\
&\geq \left|\frac{x}{4} - \frac{y}{4}\right| \\
&\geq d(Tx, Ty)
\end{aligned}$$

$i.e.\quad d(Tx, Ty) \leq \alpha.[d(x, Tx) + d(x, Ty)]$

Hence, we have, $d(Tx, Tx) < \alpha.[d(x, Tx) + d(x, Ty)] + \beta[d(y, Tx) + d(y, Ty)]$

Hence, all the conditions of theorem 3.1 are satisfied.

∴ by Theorem (3.1), T has a unique fixed point in X. The unique fixed point is $x = 0$.

5. CONCLUSIONS

In this paper, we have defined a contraction-type condition on a mapping. Mappings are defined on compact metric spaces. We obtained a unique fixed point for the mappings satisfying the contraction condition, and we have generalised some well known fixed point results on compact metric spaces. The conclusion of the contraction mapping principle is valid if we consider compact spaces instead of using complete spaces, and the conclusion of the contraction mapping principle is not valid if we consider contractive mapping instead of contraction. The mappings under consideration in this paper are continuous mappings.

6. REFERENCES

1. S. C. Malik, Savita Arora, Mathematical Analysis, *New Age International Publishers*, Fourth Edition.
2. Bhardwaj T., Rajput S.S., Choudhary S., and Yadava R.N., Some Fixed Point Theorems in Compact Metric Space, *Int. J. Math,Analysis*, 2(12), 551-555, (2008).
3. Bhatnagar P, Tenguriya A, Wadkar B.R. and Yadava R.N. Fixed Point and Common Fixed Point Theorems in Compact Metric Spaces and Pseudo Compact Tichno Spaces for Self Mappings, *International Journal of Theoretical and Applied Science*, 4(1), 8-13, (2012).
4. Edlestein M, On Fixed Point and Periodic under Contracting Mappings, *J. Land, Mathsec.*, 37, 74-79, (1961).
5. Fisher, B., A Fixed Point Theorem for Compact Metric Spaces, *Publ. Inst. Math.*, 25, 193-94, (1978).
6. Iseki, A Simple Application of Reich's Fixed Point Theorem, *Maths. Sem. Notes Kobe University*, 5(1), 75-80, (1977).
7. Kannan S., Sharma P.L., Fixed Point Mapping for Compact Metric Space.
The Mathematics Education, XXIV, (1990).

8. Namdeo R.K. and Fisher B., Related Fixed Point Theorem for Two Pairs of Set Valued Mappings on Two Complete and Compact Metric Spaces, *Indian J. Math.*, 46(2), 161- 171, (2004).
9. Sastry K.P.R., Srinivasa Rao, Fixed Point Theorems for Self-Maps on Complete and Compact Metric Spaces, *Acta Ciencia Indica*, 76 (4), 295-298, (2000).
10. Ljoubomir Ciric, Fixed-Point Mappings in Compact Metric Spaces, *De L'Institute Mathematique, Nouvelle Serie*, *Tome* 30 (44), 29-31, (1981).
11. Getahun Bekele, Alemayehu Geremew, Some Generalized Fixed Point Results on Compact Metric Spaces, *Ethiop, J. Education and Science*, 10 (1), 53-60, (2014).