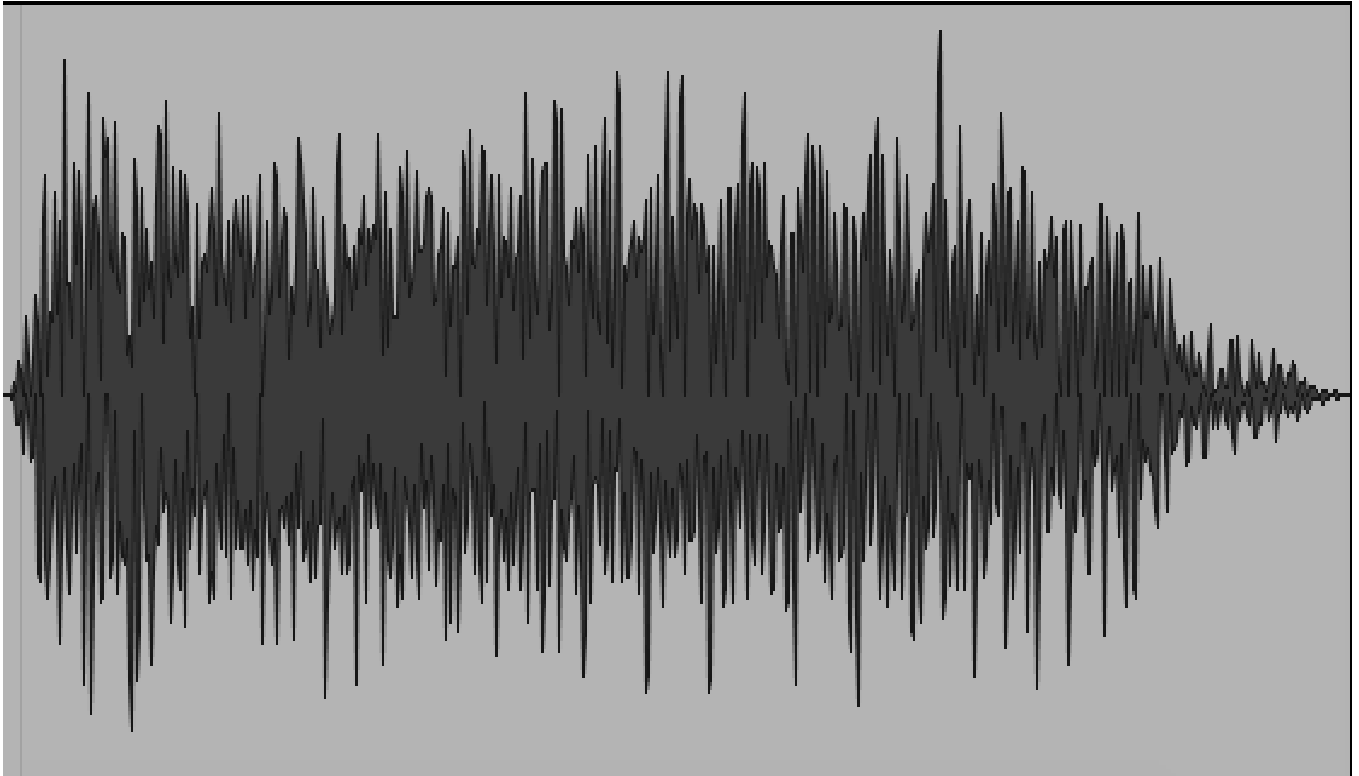


THE BEHAVIOR OF THE JUMP DISCONTINUOUS FUNCTIONS IN TIMBRE



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DEDICATION- PROJECT WORK

First of all, I thank to my parents **V FREDARIC ASSIS** (my father), **G HEMALATHA** (my mother) and **NANCY F** (my sister) encouraging me from the commencement of my project work. Without their encouragement, I cannot complete my project work.

I specially thanks to **Dr. D. FRANCIS XAVIER**, Assistant Professor, Department of Mathematics, Loyola College for helping me to clarify my doubts about the musical concepts and the behavior of such musical instruments and the musical songs.

I specially thanks to **JOHNSON VINCENT**, my friend who was doing audio engineering helps me to clarify my doubt about musical waves. And I thank to **JOSIL RAJA S** and **VINODHAN G**, Department of Mathematics, Loyola College for helping me to finish my project without them, the project will be incomplete.

And finally my friends who where encouraging me to complete my project without them the project will be incomplete.

ABSTRACT

I have decided to show you that how mathematics is related to music. The structure of every musical instruments is related to the applied topics of mathematics like, logarithms, golden ratio, etc., Many of the people don't like mathematics, but in this topic, you can practically knows about that how music is related to the mathematical concepts. We will explain each and every concept in a simple way that you can realize that our sensitivity to sound is linked to the logic of our brains. Every musical instruments have distinct sounds while playing the same frequency. Many people knows about that the music is continuous in one frequency. But in the quality of the sound, the discontinuity occurs in some musical instruments. We can see how the discontinuity occurs in such behavior of the musical waveform in one frequency.

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CHAPTER 1

INTRODUCTION

1.1 INTRODUCTION TO MATHEMATICS AND MUSIC

Early Indian and Chinese theorists show that the mathematical laws of harmonics and rhythms were fundamental not only our understanding of the world but to human well being and they studied the mathematical principles of sound. The Pythagoreans of ancient Greece were the first researchers known to have investigated the expression of musical scales in terms of numerical ratios, particularly the ratios of small integers and he regarded the small numbers 1, 2, 3, 4 as the source of all perfections.

1.2 CONTINUOUS FUNCTION

The function $f : X \rightarrow \mathbb{R}$ where $X \subset \mathbb{R}$ then the function is continuous if $\exists a \in X \exists \lim_{x \rightarrow a} f(x) = f(a)$ i.e., for $\varepsilon > 0, \exists \delta > 0 \exists |f(x) - f(a)| < \varepsilon$ whenever $|x - a| < \delta$.

In order to understand continuity, it helps to show that how a function can fail to be continuous. All of the function are used in analysis, they are continuous except at isolated points. Such points are called **points of discontinuity**.

Thus, if the function is continuous except at a point a (a is a point of discontinuity), the the types are given as follows.

1.3 REMOVABLE DISCONTINUITY

In removable discontinuity, $\lim_{x \rightarrow a} f(x)$ exists, but $\lim_{x \rightarrow a} f(x) \neq f(a)$. That is, $f(a)$ is undefined or $f(a)$ has wrong value. This discontinuity can be removed by changing the value of $f(x)$ at point a so that the new value will be $\lim_{x \rightarrow a} f(x)$.

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For example

$$f(x) = \begin{cases} \frac{x^2 - 1}{x - 1} & \text{if } x \neq 1 \\ 3 & \text{if } x = 1 \end{cases}$$

In the above function, $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^-} f(x) = 2$ but, $f(1) = 3$. The function is removable discontinuity at $x=1$

1.4 INFINITE DISCONTINUITY

In an infinite discontinuity, the one sided limit exist and atleast one of the limit should be $\pm\infty$. It is also called as asymptotic discontinuity.

For example :

$$f(x) = \begin{cases} x^2 + 2x + 2 & \text{if } x \leq -1 \\ \ln(x + 1) & \text{if } x > -1 \end{cases}$$

1.5 ESSENTIAL DISCONTINUITY

An essential discontinuity is the one in which atleast one of the one sided limits does not exist.

1.6 JUMP DISCONTINUITY

The left hand and right hand limit of a function $f(x)$ exist but not equal, ie.,

$\lim_{x \rightarrow a^+} f(x) = L_1 \neq L_2 = \lim_{x \rightarrow a^-} f(x)$ is called jump discontinuity at a point $a \in \mathbb{R}$. In a jump discontinuity, the size of the jump is the oscillation. The jump discontinuity is also called step discontinuity or discontinuity of 1st kind.

For example : Square waves in electrical engineering and musical works.

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$$f(x) = \begin{cases} x & \text{if } 0 \leq x \leq \pi \\ 2\pi - x & \text{if } \pi \leq x \leq 2\pi \end{cases}$$

1.7 PIECEWISE CONTINUOUS FUNCTION

A function $f(x)$ defined on an interval $[a, b]$ is said to be piecewise continuous if it is continuous on the interval except for a finite number of jump discontinuities. A function $f(x)$ defined on an interval $[a, b]$ is said to be **piecewise smooth** if $f(x)$ and $f'(x)$ are piecewise continuous function.

For example :

$$f(x) = \begin{cases} x - 3 & \text{if } 0 \leq x \leq \pi \\ 2\pi - x & \text{if } \pi \leq x \leq 2\pi \end{cases}$$

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CHAPTER 2

REVIEW OF LITERATURE

Music theorists, like experts in other disciplines use mathematics to develop and communicate their ideas.

2.1 ANCIENT GREEKS

Pythagoras, Plato and Aristotle were the three mathematicians detailing about the relationship and historical connection between mathematics and music. Early Greek teachers and their school of thought (i.e.,) the school of Pythagoras, Plato and Aristotle they explained not only began to study the mathematics and music but also considered that music to be part of mathematics. Ancient Greek mathematics education was divided into four sections

- i. Number Theory
- ii. Geometry
- iii. Music
- iv. Astronomy

This division of mathematics into four subtopics is called quadrivium. Until the end of the middle ages, ie., approximately 1500 A.D., in European culture, music should be studied as the part of the mathematics.

2.2 DURING 11TH CENTURY

In music, they write the notes in staff. It is similar to the mathematical graphs of discrete functions in 2-D Cartesian co-ordinates such that the x-axis denotes time and y-axis denotes the pitch which is in the staff. In fact, many musical scores (12th century), musicians have

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many forms that are similar to mathematical graphs or mathematical diagrams. Besides abstract language and notation, they use mathematical concepts like symmetry, periodicity, proportion, discreteness and continuity which are makes up the piece of music. The musical words have been studied and applied to mathematics. For example, Harmonic is a word that is used throughout mathematics like Harmonic series, but its origin is in music theory.

2.3 DURING 14TH – 17TH CENTURY

During after the rebirth, musicians were music theorists, not performers. Music research and teaching were occupations considered more prestigious than music composing or performing.

2.4 DURING 17TH AND 18TH CENTURY

Many of the most prominent and significant mathematicians were also music theorists. For example, René Descartes has many mathematical achievements include creating the field of analytical geometry and developing Cartesian geometry. He wrote his first book, *Compendium Musicale* (Compendium of music) was about music theory in 1618. In 1731, Euler published the book called *Tentamen Novae Theoriae Musicae Excertissimis Harmoniae Principiis Dilucide Expositae* (A attempt at a new theory of music, exposed in all clearness according to the most well-founded principles of harmony). D'Alembert was a French Mathematician, physicist, and philosopher who was instrumental in studying wave equations.

During 18th century, calculus was used in the discussions of vibrating strings. Brook Taylor found an equation representing the vibrations of a string based on the initial equation, and he found that the sine curve was a solution to this equation. D'Alembert was also led the differential equation of Taylor,

$$\frac{\partial^2 y}{\partial x^2} = a^2 \frac{\partial^2 y}{\partial t^2}$$

where x-axis is in the direction of the string and y-axis is the displacement at time t

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*A musical theory should be statable as connected set of axioms, definitions and theorems, the proofs of which are derived by means of an appropriate logic. – **Milton Babbitt***

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CHAPTER 3

SOUND AND ITS TYPES

Why we hear the different sounds in each different musical instruments (like piano, guitar, violin, flute, etc.,) while playing the same note or same frequency?

3.1 INTRODUCTION

Music and Mathematics plays a vital role in human daily life. To a certain extent, how music and mathematics are related. We can simply say that timbre is the one which makes one sound different from another sound. Ultimately, the concepts of timbre will depend on all begin with a closer inspection of sine waves and any sound wave can be broken into a series of simple sine waves. Mathematics and Music have been closed to each other. From harmony to number theory, to musical scales and group theory, and far beyond, mathematics and music always overlaps and connections to explore.

3.2 TIMBRE

Timbre is defined as the quality of a sound made by an each musical instruments or each voice of the humans. It is very useful in being distant in pitch, intensity, and loudness as a descriptor of sound. It depends upon the waveforms which varies with the number of overtones or harmonics, that are present in frequencies and their relative intensities. Every voice has its own timbre. Singers, especially trained singers, can change that timbre in a way that can alter the sound within the overtones and frequencies, giving a brand new sound to life, with a different color, quality and tone.

3.3 DIFFERENCE BETWEEN NOISE AND MUSIC

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In this topic, we have to discuss about the differences between noise and music. Both noise and music are not pure tone (mixture of sound with different frequencies). But music is said to be ordered tone while noise is disordered sound. In music, the waves can be separated into individual frequencies but in noise, it cannot be separated. Music has a positive effect of health but noise has not.

3.4 PURE SOUND OR PURE TONE

The waveform of pure tone is sine or sinusoidal waveform. The sine wave is clean and clear sounding. It contains only the first fundamental tone (first harmonics). For example, a pure musical note from a tuning fork is a basic sine wave.

3.5 HARMONICS OF INSTRUMENTS

The two important characteristics of a musical sound are volume and pitch of the sound. Eventhough the fundamental frequency are same, the pressure of the flute, violin, and oboe waveforms looks very different and sounds makes very different because of the harmonics or overtones of the seires along with the fundamental frequency waveforms.

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CHAPTER 4

TYPES OF WAVEFORMS

4.1 WAVEFORM AND ITS TYPES

The waveform of a signal is the graphical representation as a function of time which is independent of its time magnitude scales. The waveform of a general periodic function affects its timbre. Synthesizers are the latest technology which we can generate the sounds with many complicated waveform. There are few types of waveforms. The types of waveforms are given as follows. For smooth waveforms, the higher harmonics will be negligible.

4.1.1 PERIODIC WAVEFORMS

Waveforms which have a repeating pattern of vibrations are called periodic waveforms or it is defined as a cycle of pressure variation in a sound wave that repeats itself regularly. For example, the vocal speech of vowel sound can create periodic waveform. The degree of periodicity also varies in each vowels of natural speech due to aspiration noise and the irregular oscillation of the vocal folds that takes place. O'Connor states that "Periodic sounds give rise to a clear sensation of pitch whose height is related to the frequency of vibration, the higher the frequency, the higher the pitch. But not all periodic sounds have the simple and sinusoidal shape of the vibrations".

4.1.2 NON PERIODIC WAVEFORMS

Waveforms which does not show a repeating pattern of vibration are called non-periodic waveforms. For non-periodic sounds, there is no fundamental, no harmonics; on the contrary, noise is going on at every frequency, which is why we don't perceive any clear pitch for such sounds as for those periodic ones. For example, A person which spoke consonant sounds involves non-periodic waveforms

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4.2 TYPES OF PERIODIC WAVEFORMS

In a periodic waveform, there are several types of waveforms such as square wave, triangular or triangle wave, sawtooth wave, sine wave, cosine wave, trapezoidal wave, etc., and some of them are given as follows.

4.2.1 SQUARE WAVE

A square wave is a non-sinusoidal periodic waveform which has piecewise continuous function consists of series of only odd harmonics. The amplitude of each partial is inversely proportional to its number. It is used for creating reed instruments, pads, and basses. The harmonic structure of the square waves create a hollow sound. The function of the square wave is denoted by

$$f(x) = \begin{cases} 1 & 0 < x < \pi \\ 0 & x = 0, \pm\pi \\ -1 & -\pi < x < 2\pi \end{cases}$$

4.2.2 SAWTOOTH WAVE

A sawtooth wave is a non sinusoidal periodic waveform which has piecewise continuous function consists of series of both odd and even harmonics. It is clear and bright sounding. It is useful for creating string, pad, bass and brass sounds. The sawtooth wave has almost uniform integer harmonics on its fundamental frequency. Because sawtooth wave are quite rough, unlike sine waves, they can be deformed into various tones by removing the harmonics using filters. The function of the sawtooth wave is denoted by

$$f(x) = \begin{cases} x & 0 \leq x \leq \pi \\ x - 2\pi & \pi \leq x \leq 2\pi \end{cases}$$

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4.2.3 TRIANGULAR WAVE

A triangular wave is a continuous function but not differentiable i.e., it has no jump discontinuities, but its derivatives changes discontinuity twice per cycle. It is a non-sinusoidal periodic waveform which has piecewise linear function but has continuous on real intervals. Like a square wave, the triangular wave also consists of a series of only odd harmonics. The function of the triangular wave is denoted by

$$f(x) = \begin{cases} \frac{x-a}{b-a} & a < x < b \\ 0 & c \leq x \leq a \\ \frac{c-x}{c-b} & b < x < c \end{cases}$$

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CHAPTER 5

FOURIER SERIES

5.1 INTRODUCTION TO FOURIER SERIES

Most of the waveforms occurs in practice are continuous and single valued . However when sudden changes like square waveforms, etc., theoretically vertical lines could occur in the waveform which gives multi-values at instant. As long as these multi-values occur over finite bounds, the waveform is single valued and continuous in pieces, or it is said to be piecewise continuous.

Fourier theorem states that, “Any periodic oscillation curve with frequency f can be broken up analyzed into a set of simple sine curves of frequencies $f, 2f, 3f, \dots$ each with its own amplitude.” The function f be a piecewise continuous function $[-\pi, \pi]$. Then the fourier series of the function f is of the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

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$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

Are called Fourier co-efficient of the function f . In order to analyzing the sounds of musical instruments, Fourier series enable us to synthesize sounds. The idea behind the music synthesizers is that we can combine the various pure tones to create a clear and richer sound by adding certain harmonics.

5.2 GIBBS PHENOMENON

If the given function f has jump discontinuity, then the partial sum of the Fourier series has oscillates near the jump, which might increases the maximum of the partial sum above the function. This phenomenon is called Gibbs phenomenon. That is, the undershooting and the overshooting of the series which is near the discontinuities is called Gibbs phenomenon. For example, the overshoot of a function in a sawtooth wave at a jump 1 is approximately equals 9% in Gibbs phenomenon.

5.3 FOURIER CONVERGENCE THEOREM

In a piecewise continuous function, if the function f is continuous then the fourier series converges to $f(x)$ at all points and if the function f is discontinuous at a point x_0 then the fourier series converges to $\frac{f(x_0+) + f(x_0-)}{2}$. Thus a Fourier series converges to the average value of the left and right limit at a point of discontinuity of the function $f(x)$. This theorem is called Fourier convergence theorem.

If $f(x)$ is a piecewise smooth function on the interval $[-\pi, \pi]$, then for any $x_0 \in [-\pi, \pi]$

$$\lim_{N \rightarrow \infty} f_N(x_0) = \begin{cases} f(x_0) & f(x) \text{ is continuous on } [-\pi, \pi] \\ \frac{f(x_0+) + f(x_0-)}{2} & f(x) \text{ has a jump discontinuity at } x_0 \end{cases}$$

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Where $f(x_0^+)$ represents the right limit and $f(x_0^-)$ represents the left limit at a point x_0 . There are certain jump discontinuous function which generates the fourier series waveform converges at the finite number of jump discontinuities, we have to solve the proofs by generating the fourier series waveforms of certain discontinuous waves.

5.4 SQUARE WAVE FUNCTION USING FOURIER SERIES

The function of the square wave can be expressed as

$$f(x) = \begin{cases} 1 & 0 < x < \pi \\ 0 & x = 0, \pm\pi \\ -1 & -\pi < x < 2\pi \end{cases} \quad \dots\dots\dots (5.4.1)$$

The formula for Fourier series is given as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \dots\dots\dots (5.4.2)$$

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_0^{2\pi} f(x) dx \\ &= \frac{1}{\pi} \left[\int_0^{\pi} dx - \int_{\pi}^{2\pi} dx \right] \\ &= \frac{1}{\pi} [(\pi - 0) - (2\pi - \pi)] \\ &= \frac{1}{\pi} [\pi - \pi] \\ a_0 &= 0 \quad \dots\dots\dots (5.4.3) \end{aligned}$$

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$$\begin{aligned}
 a_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx \\
 &= \frac{1}{\pi} \left[\int_0^{\pi} \cos nx \, dx - \int_{\pi}^{2\pi} \cos nx \, dx \right] \\
 &= \frac{1}{\pi} [(0 - 0) + (0 - 0)] \\
 a_n &= 0 \quad \dots\dots\dots (5.4.4)
 \end{aligned}$$

$$\begin{aligned}
 b_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx \\
 &= \frac{1}{\pi} \left[\int_0^{\pi} \sin nx \, dx - \int_{\pi}^{2\pi} \sin nx \, dx \right] \\
 &= \frac{1}{\pi} \left[\frac{-(-1)^n}{n} + \frac{1}{n} + \frac{1}{n} - \frac{(-1)^n}{n} \right] \\
 &= \frac{1}{\pi} \left[\frac{2}{n} - \frac{2(-1)^n}{n} \right]
 \end{aligned}$$

$$b_n = \frac{2}{\pi n} (1 - (-1)^n) \quad \dots\dots\dots (5.4.5)$$

Substitute (5.4.3), (5.4.4), (5.4.5) in (5.4.2), we get,

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$$f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(1 - (-1)^n)}{n} \sin nx$$

$$f(x) = \frac{4}{\pi} \left(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \cdots \dots \dots \right) \dots\dots\dots (5.4.6)$$

$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin(2n-1)x$$

Which is the required Fourier series of the square wave. But in square wave,

$$\lim_{x \rightarrow 0^+} f(x) = f(0 + \varepsilon) = 1 \quad \text{and} \quad \lim_{x \rightarrow 0^-} f(x) = f(0 - \varepsilon) = -1$$

$$\lim_{x \rightarrow 0^+} f(x) < \infty, \lim_{x \rightarrow 0^-} f(x) < \infty \text{ but } \lim_{x \rightarrow 0^+} f(x) \neq \lim_{x \rightarrow 0^-} f(x)$$

Therefore, there is a jump discontinuous function at $x=0$,

Similarly,

$$\lim_{x \rightarrow \pi^+} f(x) = f(\pi + \varepsilon) = -1 \quad \text{and} \quad \lim_{x \rightarrow \pi^-} f(x) = f(\pi - \varepsilon) = 1$$

$$\lim_{x \rightarrow \pi^+} f(x) < \infty, \lim_{x \rightarrow \pi^-} f(x) < \infty \text{ but } \lim_{x \rightarrow \pi^+} f(x) \neq \lim_{x \rightarrow \pi^-} f(x)$$

Therefore, there is a jump discontinuity at $x = \pm\pi$,

For jump discontinuous function, The Fourier convergence theorem of the square wave is given by,

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$$\frac{\lim_{x \rightarrow 0^+} f(x) + \lim_{x \rightarrow 0^-} f(x)}{2} = \frac{1 + (-1)}{2} = 0$$

Similarly,

$$\frac{\lim_{x \rightarrow \pi^+} f(x) + \lim_{x \rightarrow \pi^-} f(x)}{2} = \frac{(-1) + 1}{2} = 0$$

Thus, the Fourier series of the function $f(x)$ at $x=0, \pm n\pi$ will converge to the point zero.

5.5 SAWTOOTH WAVE FUNCTION USING FOURIER SERIES

The function can be expressed as,

$$f(x) = \begin{cases} x & 0 \leq x \leq \pi \\ x - 2\pi & \pi \leq x \leq 2\pi \end{cases} \quad \dots\dots\dots (5.5.1)$$

The formula for Fourier series is given as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \dots\dots\dots (5.5.2)$$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx$$

$$= \frac{1}{\pi} \left[\int_0^{\pi} x dx + \int_{\pi}^{2\pi} x dx - 2\pi \int_{\pi}^{2\pi} dx \right]$$

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$$= \frac{1}{\pi} \left[\left(\frac{\pi^2}{2} - 0 \right) + \left(\frac{4\pi^2}{2} - \frac{\pi^2}{2} \right) - 2\pi(2\pi - \pi) \right]$$

$$= \frac{1}{\pi} \left[\frac{\pi^2}{2} + \frac{3\pi^2}{2} - 2\pi^2 \right] = 0$$

$$a_0 = 0 \quad \dots\dots\dots (5.5.3)$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi} \left[\int_0^{\pi} x \cos nx \, dx + \int_{\pi}^{2\pi} x \cos nx \, dx - 2\pi \int_{\pi}^{2\pi} \cos nx \, dx \right]$$

$$= \frac{1}{\pi} \left[-\frac{1}{n^2} - \frac{1}{n^2} + \frac{1}{n^2} + \frac{1}{n^2} \right] = 0$$

$$a_n = 0 \quad \dots\dots\dots (5.5.4)$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx$$

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$$= \frac{1}{\pi} \left[\int_0^{\pi} x \sin nx \, dx + \int_{\pi}^{2\pi} x \sin nx \, dx - 2\pi \int_{\pi}^{2\pi} \sin nx \, dx \right]$$

$$= \frac{1}{\pi} \left[\frac{-\pi(-1)^n}{n} - \frac{-\pi(-1)^n}{n} \right]$$

$$b_n = \frac{-2(-1)^n}{n} \dots\dots\dots (5.5.5)$$

Substitute (5.5.3), (5.5.4), (5.5.5), in (5.5.2) we get,

$$f(x) = -2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx$$

Which is the required Fourier series of sawtooth wave.

But in Sawtooth wave,

$$\lim_{x \rightarrow \pi^+} f(x) = f(\pi + \varepsilon) = (\pi + \varepsilon) - 2\pi = -\pi + \varepsilon$$

when $\varepsilon \rightarrow 0$,

$$\lim_{x \rightarrow \pi^+} f(x) = -\pi \quad \text{and}$$

$$\lim_{x \rightarrow \pi^-} f(x) = f(\pi - \varepsilon) = \pi - \varepsilon$$

when $\varepsilon \rightarrow 0$,

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$$\lim_{x \rightarrow \pi^-} f(x) = \pi$$

$$\lim_{x \rightarrow 0^+} f(x) < \infty, \lim_{x \rightarrow 0^-} f(x) < \infty \quad \text{but} \quad \lim_{x \rightarrow 0^+} f(x) \neq \lim_{x \rightarrow 0^-} f(x)$$

Therefore, there is a jump discontinuous function at $x = \pm\pi$,

For jump discontinuous function, the Fourier convergence theorem of sawtooth wave is given by,

$$\frac{\lim_{x \rightarrow \pi^+} f(x) + \lim_{x \rightarrow \pi^-} f(x)}{2} = \frac{(-\pi) + \pi}{2} = 0$$

Thus, the Fourier series of the function $f(x)$ at $x = \pm\pi$ will converge to the point zero.

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CHAPTER 6

BEHAVIOR OF WAVEFORMS IN TIMBRE

By using the waveform pattern, we can generate the new musical instrument sound but not natural (For example : We can generate the piano sound, but not the natural sound of piano). What should we proceed to produce the natural sound?

6.1 DIMENSIONS IN TIMBRE

This is an approach that has been important in the perceptual literature that has been to scale the perceived degree of relationship between the sounds with different timbres. All possible pairs of timbres are presented to listeners, and for each pair, they are required to rate how similar or dissimilar the two timbres to each other. The resulting dissimilarity ratings are analyzed using a method called multi-dimensional scaling.

According to Rasch and Plomp (1999), they suggest that timbre has two dimensions. Gordon and Grey (1978), Grey (1977), Marozeau et al (2003), McAdams et al (1995) and Plomp (1970) suggest that they have three dimensions. Von Bismark (1974) and Stepanek (2006) suggest that they have four dimensional space. Now, we use the four dimensional space (four parameters). The parameters are shown as below.

6.2 FOUR DIMENSIONAL TIMBRE SPACE

6.2.1 THE SPECTRUM PARAMETER

It controls the base, unfiltered spectrum of the sound. Traversing the spectrum values largely follows a transition from a sine wave to a square wave to a sawtooth wave. (i.e.,) It starts with the fundamental frequency ($\sin x$) gradually adding odd harmonics ($\sin 3x$, $\sin 5x$, $\sin 7x$,....) and then adding even ones ($\sin 2x$, $\sin 4x$, $\sin 6x$,....).

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The spectra parameter ranges from 0-255. In 0, it results sine wave, in 127, it results square wave and in 255, it results sawtooth wave. The non-sinusoidal wave ranges from 120-255 have jump discontinuous function.

6.2.2 THE BRIGHTNESS PARAMETER

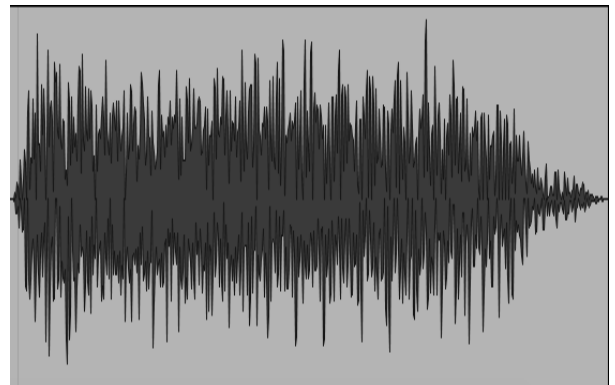
The brightness is experimentally correlated to spectral centroid is recreated as a frequency filter. The filter is either low pass or high pass which depends upon the parameters.

6.2.3 THE ARTICULATION PARAMETER

It controls the initial spectro temporal evolution of the timbre, accounting for the asynchrony in the rise of harmonics. Its implementation is a filter with changing the cut-off frequency, where the rate of change of the cut-off is affected by the parameters distance from the neutral centre.

6.2.4 THE ENVELOPE PARAMETER

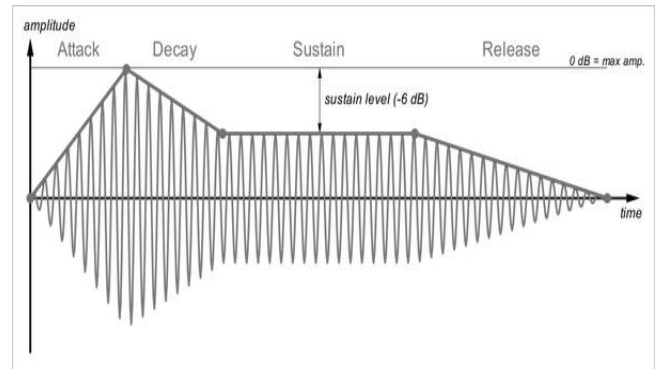
It controls the temporal amplitude envelope, based on the attack, decay, sustain, and release (ADSR) paradigm. The value of envelope primarily affects the attack time. At low envelope values, attack time is at a minimum to emulate percussive sounds, and at a maximum at high values for sounds with softer onsets. Altogether, these dimensions as listed to form a consecutive signal chain. It starts as a waveform determined by the spectrum, which is then filtered by the brightness and then the articulation filters and the final effect is a gain that follows the envelope's ADSR profile. The result is an "wide range of sounds and recognizably mimicking various instruments".



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6.2.4.1 ADSR-ATTACK

The attack determined as long as the sound produced as a full volume while pressing keys. This is the first factor of the sound and it will take time to decay.



6.2.4.2 ADSR-DECAY

After the peak of sound of attack, the length of time the note that reach the sustained level. This is often used to change the overall of the sound track.

6.2.4.3 ADSR-SUSTAIN

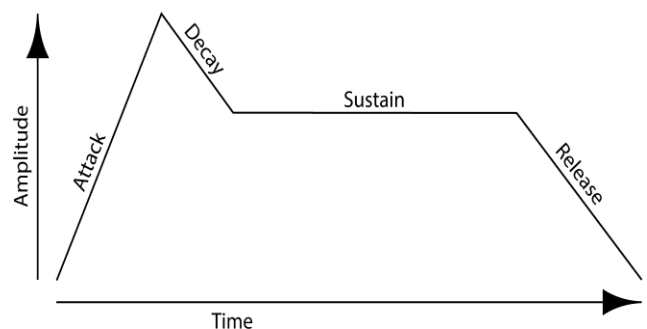
Sustain determines the volume of the sound takes time between decay and release. Apart from the sound of attack, it determines the overall volume of the sound.

6.2.4.4 ADSR-RELEASE

Release determines the end of the sound. It allows to see how long the sound takes after releasing the key. These are the four paradigm of envelope.

6.3 SOUND PRODUCED BY SQUARE WAVE

Square wave is one of the non sinusoidal piecewise continuous waves. It has finite number of jump discontinuities. In nature, each musical instruments have different waves while playing different



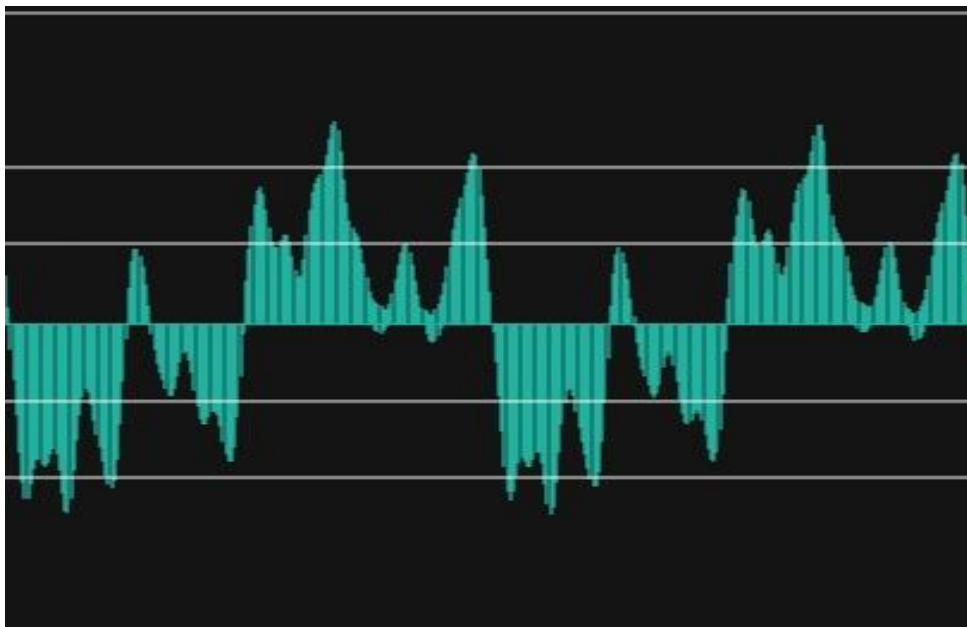
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frequencies. For example, the saxophone instruments sounds like sawtooth in 440 Hz but, it sounds pure sine wave while playing higher frequency. In synthesizer, we can tune certain frequency to produce different musical instruments by using the periodic waves. In square waves, we can produce such musical instruments called clarinet. A clarinet is a cylindrical bore instrument closed at one end. Hence, the normal resonant modes must have a pressure maximum near the first open key. In clarinet waveform, the sound looks like square wave. The Fourier series of clarinet instrument is

$$f(x) = \sin wx + 0.75 \sin 3wx + 0.5 \sin 5wx + 0.14 \sin 7wx + 0.5 \sin 9wx + 0.12 \sin 11wx + 0.17 \sin 13wx$$

where w determines frequency. The series has only odd harmonics. But in Fourier Series, the square wave (in (5.4.6)) series also has only odd harmonics but the co-efficient changes.

We already know that, Changing the co-efficient of continuous function is continuous function. The waveform of a clarinet wave is given as



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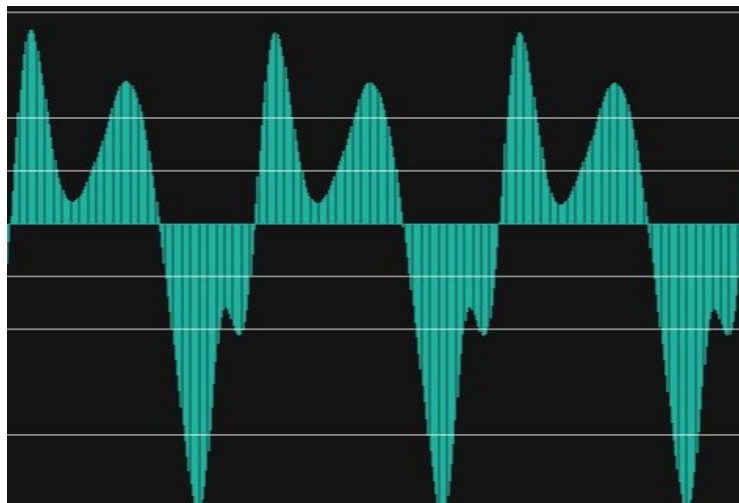
The square wave has jump discontinuities and the function converges to 0 and $\pm\pi$. While changing co-efficients of square wave resultants jump discontinuity. Thus, the clarinet also has jump discontinuous functions. Due to the dimension of timbre, the sound looks rich. But the sound looks like square wave. In clarinet, the spectra parameter has 126, Brightness parameter has 90, Articulation parameter has 111 and Envelope parameter has 206.

6.4 SOUND PRODUCED BY SAWTOOTH WAVE

In sawtooth wave, we can produce and generate such musical instruments called saxophone. In saxophone, the sound looks like sawtooth wave. We can produce the sawtooth wave into different types of saxophone and other different musical instruments. The Fourier series of the sawtooth wave is

$$f(x) = \sin wx + 0.38 \sin 2wx + 0.14 \sin 3wx + 0.02 \sin 4wx$$

where w determines the frequency of the saxophone. The series has both odd and even harmonics but the series is closely to sawtooth waveform. The waveform of the saxophone is given as follows



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The sawtooth wave has jump discontinuities and the function converges to $\pm\pi$. While changing co-efficients of sawtooth wave resultants jump discontinuous function. Thus, the sound of saxophone also has jump discontinuous functions. Due to the dimension of timbre, the sound looks rich. But the sound looks like sawtooth wave.

6.5 LIST OF MUSICAL INSTRUMENT WHICH HAS JUMP DISCONTINUITY

We have already told that increasing the range of spectra parameter, the jump discontinuity appears. Musical instruments like piano, bass, violin, clarinet, saxophone, etc. appears jump discontinuous functions. The Fouier series of such instruments converge to the points which has lack of continuous.

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CHAPTER 7

CONCLUSION AND FUTURE RESEARCH

7.1 CONCLUSION

Timothy Weiss questioned “Does jump discontinuity have something to do with timbre?”. This is my inspiration. This topic tells about that in what ways the musical sound differs and we know that the music has continuous wave, but I questioned whether discontinuity appears in music or not. And I concluded that while using synthesizer, we can create such musical instrument sound using different types of non-sinusoidal wave and the corresponding four parameters. I find the jump discontinuous function using periodic function in musical instruments. Because, we can determine the frequencies in periodic function. So, my conclusion is there are such musical instrument which has jump discontinuity by using periodic function.

7.2 FUTURE RESEARCH

In mathematics, we have dimensions and spaces, but in music, we don't have dimensions properly and we can find the timbre space while using mathematical concepts. Does the jump discontinuity appears in non-periodic functions? These are the future research of the topic Mathematics and Music.

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